1. Mathematics — Textbooks, 1811
2. Mechanics — " "

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A

COURSE

OF

MATHEMATICS.

IN THREE VOLUMES.

COMPOSED FOR

THE USE OF THE ROYAL MILITARY ACADEMY,

BY ORDER OF HIS LORDSHIP

THE MASTER GENERAL OF THE ORDNANCE.

BY

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THE SIXTH EDITION,

ENLARGED AND CORRECTED.

VOL. I.

LONDON:

PRINTED FOR F. C. AND J. RIVINGTON; G. WILKIE AND J. ROBINSON; J. WALKER; G. ROBINSON; LACKINGTON, ALLEN, AND CO.; VERNOR, HOOD, AND SHARPE; G. KEARSLEY; LONGMAN, HURST, REES, ORME, AND BROWN; CADELL AND DAVIES; J. CUTHELL; B. CROSBY AND CO.; J. RICHARDSON; J. M. RICHARDSON; BLACK, PARRY, AND KINGSBURY; GALE AND CURTIS; AND J. JOHNSON AND CO.

1811.

T. Davison, Lombard Street,
Whitefriars, London.

PREFACE.

A SHORT and Easy Course of the Mathematical Sciences has long been considered as a desideratum for the use of Students in the different schools of education: one that should hold a middle rank between the more voluminous and bulky collections of this kind, and the mere abstract and brief common-place forms, of principles and memorandums.

For long experience, in all Seminaries of Learning, has shown, that such a work was very much wanted, and would prove a great and general benefit; as, for want of it, recourse has always been obliged to be had to a number of other books, by different authors; selecting a part from one and a part from another, as seemed most suitable to the purpose in hand, and rejecting the other parts—a practice which occasioned much expence and trouble, in procuring and using such a number of odd volumes, of various forms and modes of composition; besides wanting the benefit of uniformity and reference, which are found in a regular series of composition.

To remove these inconveniences, the Author of the present work has been induced, from time to time, to compose various parts of this Course of Mathematics; which the experience of many years' use in the Academy has enabled him to adapt and improve to the most useful form and quantity, for the benefit of instruction there. And, to render that benefit more eminent and lasting, the Master General of the Ordnance has been pleased to give it its present form, by ordering it to be enlarged and printed, for the use of the Rcayl Military Academy.

As this work has been composed expressly with the intention of adapting it to the purposes of academical education, it is not designed to hold out the expectation of an entire new mass of inventions and discoveries: but rather to collect and arrange the most useful known principles of mathematics, disposed in a convenient practical form, demonstrated in a plain and concise way, and illustrated with suitable examples; rejecting whatever seemed to be matters of mere curiosity, and retaining only such parts and branches, as have a direct tendency and application to some useful purpose in life or profession.

It is however expected that much that is new will be found in many parts of these volumes; as well in the matter, as in the arrangement and manner of demonstration, throughout the whole work, especially in the geometry, which is rendered much more easy and simple than heretofore; and in the conic-sections, which are here treated in a manner at once new, easy, and natural; so much so indeed, that all the propositions and their demonstrations, in the ellipsis, are the very same, word for word, as those in the hyperbola, using only, in a very few places, the word *sum*, for the word *difference:* also in many of the mechanical and philosophical parts which follow, in the second volume. In the conic sections, too, it may be observed, that the first theorem of each section only is proved from the cone itself, and all the rest of the theorems are deduced from the first, or from each other, in a very plain and simple manner.

Besides renewing most of the rules, and introducing everywhere new examples, this edition is much enlarged in several places; particularly by extending the tables of squares and cubes, square roots and cube roots, to 1000 numbers, which will be found of great use in many calculations; also by the table of logarithms at the end of the first volume, and of logarithms, sines, and tangents, at the end of the second volume; by the addition of Cardan's rules for resolving cubic equations;

equations; with tables and rules for annuities; and many other improvements in different parts of the work.

Though the several parts of this course of mathematics are ranged in the order naturally required by such elements, yet students may omit any of the particulars that may be thought the least necessary to their several purposes; or they may study and learn various parts in a different order from their present arrangement in the book, at the discretion of the tutor. So, for instance, all the notes at the foot of the pages may be omitted, as well as many of the rules; particularly the 1st or Common Rule for the Cube Root, p. 85, may well be omitted, being more tedious than useful. Also the chapters on Surds and Infinite Series, in the Algebra: or these might be learned after Simple Equations. Also Compound Interest and Annuities at the end of the Algebra. Also any part of the Geometry, in vol. 1; any of the branches in vol. 2, at the discretion of the preceptor. And, in any of the parts, he may omit some of the examples, or he may give more than are printed in the book; or he may very profitably vary or change them, by altering the numbers occasionally.—As to the quantity of writing; the author would recommend, that the student copy out into his fair book no more than the chief rules which he is directed to learn off by rote, with the work of one example only to each rule, set down at full length: omitting to set down the work of all the other examples, how many soever he may be directed to work out upon his slate or waste paper.—In short, a great deal of the business, as to the quantity and order and manner, must depend on the judgment of the discreet and prudent tutor or director.

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LOGARITHMS.

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GEOMETRY.

A

COURSE

OF

MATHEMATICS, &c.

GENERAL PRINCIPLES.

1. QUANTITY, or MAGNITUDE, is any thing that will admit of increase or decrease; or that is capable of any sort of calculation or mensuration: such as numbers, lines, space, time, motion, weight.

2. MATHEMATICS is the science which treats of all kinds of quantity whatever, that can be numbered or measured.—That part which treats of numbering is called *Arithmetic*; and that which concerns measuring, or figured extension, is called *Geometry*.—These two, which are conversant about multitude and magnitude, being the foundation of all the other parts, are called *Pure* or *Abstract Mathematics*; because they investigate and demonstrate the properties of abstract numbers and magnitudes of all sorts. And when these two parts are applied to particular or practical subjects, they constitute the branches or parts called *Mixed Mathematics*.—Mathematics is also distinguished into *Speculative* and *Practical*: viz. *Speculative*, when it is concerned in discovering properties and relations; and *Practical*, when applied to practice and real use concerning physical objects.

3. In Mathematics are several general terms or principles; such as, Definitions, Axioms, Propositions, Theorems, Problems, Lemmas, Corollaries, Scholiums, &c.

4. *A Definition* is the explication of any term or word in a science; showing the sense and meaning in which the term is employed.—Every Definition ought to be clear, and expressed in words that are common and perfectly well understood.

5. *A Proposition* is something proposed to be proved, or something required to be done; and is accordingly either a Theorem or a Problem.

6. *A Theorem* is a demonstrative proposition; in which some property is asserted, and the truth of it required to be proved. Thus, when it is said that, The sum of the three angles of any triangle is equal to two right angles, this is a Theorem, the truth of which is demonstrated by Geometry. —A set or collection of such Theorems constitutes a *Theory*.

7. *A Problem* is a proposition or a question requiring something to be done; either to investigate some truth or property, or to perform some operation. As, to find out the quantity or sum of all the three angles of any triangle, or to draw one line perpendicular to another.——*A Limited Problem* is that which has but one answer or solution. *An Unlimited Problem* is that which has innumerable answers. And a *Determinate Problem* is that which has a certain number of answers.

8. *Solution* of a Problem, is the resolution or answer given to it. *A Numerical* or *Numeral Solution*, is the answer given in numbers. *A Geometrical Solution*, is the answer given by the principles of Geometry. And a *Mechanical Solution*, is one which is gained by trials.

9. *A Lemma* is a preparatory proposition, laid down in order to shorten the demonstration of the main proposition which follows it.

10. *A Corollary*, or *Consectary*, is a consequence drawn immediately from some proposition or other premises.

11. *A Scholium* is a remark or observation made by some foregoing proposition or premises.

12. *An Axiom*, or *Maxim*, is a self-evident proposition; requiring no formal demonstration to prove the truth of it; but is received and assented to as soon as mentioned. Such as, The whole of any thing is greater than a part of it; or, The whole is equal to all its parts taken together: or, Two quantities that are each of them equal to a third quantity, are equal to each other.

13. *A*

13. *A Postulate*, or *Petition*, is something required to be done, which is so easy and evident that no person will hesitate to allow it.

14. *An Hypothesis* is a supposition assumed to be true, in order to argue from, or to found upon it the reasoning and demonstration of some proposition.

15. *Demonstration* is the collecting the several arguments and proofs, and laying them together in proper order, to show the truth of the proposition under consideration.

16. *A Direct, Positive*, or *Affirmative Demonstration*, is that which concludes with the direct and certain proof of the proposition in hand.—This kind of Demonstration is most satisfactory to the mind; for which reason it is called sometimes an *Ostensive Demonstration*.

17. *An Indirect*, or *Negative Demonstration*, is that which shows a proposition to be true, by proving that some absurdity would necessarily follow if the proposition advanced were false. This is also sometimes called *Reductio ad Absurdum;* because it shows the absurdity and falsehood of all suppositions contrary to that contained in the proposition.

18. *Method* is the art of disposing a train of arguments in a proper order, to investigate either the truth or falsity of a proposition, or to demonstrate it to others when it has been found out.—This is either Analytical or Synthetical.

19. *Analysis*, or the *Analytic Method*, is the art or mode of finding out the truth of a proposition, by first supposing the thing to be done, and then reasoning back, step by step, till we arrive at some known truth.—This is also called the *Method of Invention*, or *Resolution;* and is that which is commonly used in Algebra.

20. *Synthesis*, or the *Synthetic Method*, is the searching out truth, by first laying down some simple and easy principles, and pursuing the consequences flowing from them till we arrive at the conclusion.—This is also called the *Method of Composition;* and is the reverse of the Analytic method, as this proceeds from known principles to an unknown conclusion; while the other goes in a retrograde order, from the thing sought, considered as if it were true, to some known principle or fact. And therefore, when any truth has been found out by the Analytic method, it may be demonstrated by a process in the contrary order, by Synthesis.

ARITHMETIC.

ARITHMETIC is the art or science of numbering; being that branch of Mathematics which treats of the nature and properties of numbers.—When it treats of whole numbers, it is called *Vulgar*, or *Common Arithmetic*; but when of broken numbers, or parts of numbers, it is called *Fractions*.

Unity, or an *Unit*, is that by which every thing is called one; being the beginning of number; as, one man, one ball, one gun.

Number is either simply one, or a compound of several units; as, one man, three men, ten men.

An *Integer*, or *Whole Number*, is some certain precise quantity of units; as, one, three, ten.—These are so called as distinguished from *Fractions*, which are broken numbers, or parts of numbers; as, one-half, two-thirds, or three-fourths.

NOTATION AND NUMERATION.

NOTATION, or NUMERATION, teaches to denote or express any proposed number, either by words or characters; or to read and write down any sum or number.

The numbers in Arithmetic are expressed by the following ten digits, or Arabic numeral figures, which were introduced into Europe by the Moors, about eight or nine hundred years since; viz. 1 one, 2 two, 3 three, 4 four, 5 five, 6 six, 7 seven, 8 eight, 9 nine, 0 cipher, or nothing. These characters or figures were formerly all called by the general name of *Ciphers*; whence it came to pass that the art of Arithmetic was then often called *Ciphering*. Also the first nine are called *Significant Figures*, as distinguished from the cipher, which is of itself quite insignificant.

Besides this value of those figures, they have also another, which depends on the place they stand in when joined together; as in the following table:

Uuits

&c.	Hundreds of Millions	Tens of Millions	Millions	Hundreds of Thousands	Tens of Thousands	Thousands	Hundreds	Tens	Units
&c.	9	8	7	6	5	4	3	2	1
		9	8	7	6	5	4	3	2
			9	8	7	6	5	4	3
				9	8	7	6	5	4
					9	8	7	6	5
						9	8	7	6
							9	8	7
								9	8
									9

Here, any figure in the first place, reckoning from right to left, denotes only its own simple value; but that in the second place, denotes ten times its simple value; and that in the third place, a hundred times its simple value; and so on: the value of any figure, in each successive place, being always ten times its former value.

Thus, in the number 1796, the 6 in the first place denotes only six units, or simply six; 9 in the second place signifies nine tens, or ninety; 7 in the third place, seven hundred; and the 1 in the fourth place, one thousand: so that the whole number is read thus, one thousand seven hundred and ninety-six.

As to the cipher, 0, though it signify nothing of itself, yet being joined on the right-hand side to other figures, it increases their value in the same ten-fold proportion: thus, 5 signifies only five; but 50 denotes 5 tens, or fifty; and 500 is five hundred; and so on.

For the more easily reading of large numbers, they are divided into periods and half-periods, each half-period consisting of three figures; the name of the first period being units; of the second, millions; of the third, millions of millions, or bi-millions, contracted to billions: of the fourth, millions of millions of millions, or tri-millions, contracted to trillions, and so on. Also the first part of any period is so many units of it, and the latter part so many thousands.

The

The following Table contains a summary of the whole doctrine.

Periods.	Quadrill.;		Trillions;		Billions;		Millions;		Units.	
Half-per.	th.	un.	th.	un.	th.	un.	th.	un.	th.	un.
Figures.	123,	456;	789,	098;	765,	432;	101,	234;	567,	890.

Numeration is the reading of any number in words that is proposed or set down in figures; which will be easily done by help of the following rule, deduced from the foregoing tablets and observations—viz.

Divide the figures in the proposed number, as in the summary above, into periods and half-periods; then begin at the left-hand side, and read the figures with the names set to them in the two foregoing tables.

EXAMPLES.

Express in words the following numbers; viz.

34	15080	13405670
96	72003	47050023
180	109026	309025600
304	483500	4723507689
6134	2500639	274856390000
9028	7523000	6578600307024

Notation is the setting down in figures any number proposed in words; which is done by setting down the figures instead of the words or names belonging to them in the summary above; supplying the vacant places with ciphers where any words do not occur.

EXAMPLES.

Set down in figures the following numbers;

Fifty-seven.
Two hundred eighty-six.
Nine thousand two hundred and ten.
Twenty-seven thousand five hundred and ninety-four.
Six hundred and fortythousand, four hundred and eighty-one.
Three millions, two hundred sixty thousand, one hundred and six.

Four

Four hundred and eight millions, two hundred and fifty-five thousand, one hundred and ninety-two.

Twenty-seven thousand and eight millions, ninety-six thousand two hundred and four.

Two hundred thousand and five hundred and fifty millions, one hundred and ten thousand, and sixteen.

Twenty-one billions, eight hundred and ten millions, sixty-four thousand, one hundred and fifty.

Of the Roman Notation.

The Romans, like several other nations, expressed their numbers by certain letters of the alphabet. The Romans used only seven numeral letters, being the seven following capitals: viz. I for *one;* V for *five;* X for *ten;* L for *fifty;* C for an *hundred;* D for *five hundred;* M for a *thousand.* The other numbers they expressed by various repetitions and combinations of these, after the following manner:

1	=	I	
2	=	II	As often as any character is repeated, so many times is its value repeated.
3	=	III	
4	=	IIII or IV	A less character before a greater diminishes its value.
5	=	V	
6	=	VI	A less character after a greater increases its value.
7	=	VII	
8	=	VIII	
9	=	IX	
10	=	X	
50	=	L	
100	=	C	
500	=	D or IƆ	For every Ɔ annexed, this becomes 10 times as many.
1000	=	M or CIƆ	For every C and Ɔ, placed one at each end, it becomes 10 times as much.
2000	=	MM	
5000	=	$\overline{\mathrm{V}}$ or IƆƆ	A bar over any number increases it 1000 fold.
6000	=	$\overline{\mathrm{VI}}$	
10000	=	$\overline{\mathrm{X}}$ or CCIƆƆ	
50000	=	$\overline{\mathrm{L}}$ or IƆƆƆ	
60000	=	$\overline{\mathrm{LX}}$	
100000	=	$\overline{\mathrm{C}}$ or CCCIƆƆƆ	
1000000	=	$\overline{\mathrm{M}}$ or CCCCIƆƆƆƆ	
2000000	=	$\overline{\mathrm{MM}}$	
&c.		&c.	

EXPLA-

Explanation of certain Characters.

There are various characters or marks used in Arithmetic, and Algebra, to denote several of the operations and propositions; the chief of which are as follow:

$+$ signifies *plus*, or addition.
$-$ - - *minus*, or subtraction.
$\times$ or . - multiplication.
$\div$ - - division.
: :: : - proportion.
$=$ - - equality.
$\surd$ - - square root.
$\sqrt[3]{}$ - - cube root, &c.
$\backsim$ - - diff. between two numbers when it is not known which is the greater.

Thus,
$5 + 3$, denotes that 3 is to be added to 5.
$6 - 2$, denotes that 2 is to be taken from 6.
7×3, or $7 . 3$, denotes that 7 is to be multiplied by 3.
$8 \div 4$, denotes that 8 is to be divided by 4.
$2 : 3 :: 4 : 6$, shows that 2 is to 3 as 4 is to 6.
$6 + 4 = 10$, shows that the sum of 6 and 4 is equal to 10.
$\sqrt{3}$, or $3^{\frac{1}{2}}$, denotes the square root of the number 3.
$\sqrt[3]{5}$, or $5^{\frac{1}{3}}$, denotes the cube root of the number 5.
7^2, denotes that the number 7 is to be squared.
8^3, denotes that the number 8 is to be cubed.
&c.

OF ADDITION.

Addition is the collecting or putting of several numbers together, in order to find their *sum*, or the total amount of the whole. This is done as follows:

Set or place the numbers under each other, so that each figure may stand exactly under the figures of the same value, that

that is, units under units, tens under tens, hundreds under hundreds, &c. and draw a line under the lowest number, to separate the given numbers from their sum, when it is found. —Then add up the figures in the column or row of units, and find how many tens are contained in that sum.—Set down exactly below, what remains more than those tens, or if nothing remains, a cipher, and carry as many ones to the next row as there are tens.—Next add up the second row, together with the number carried, in the same manner as the first. And thus proceed till the whole is finished, setting down the total amount of the last row.

To prove Addition.

First Method.—Begin at the top, and add together all the rows of numbers downwards; in the same manner as they were before added upwards; then if the two sums agree, it may be presumed the work is right.—This method of proof is only doing the same work twice over, a little varied.

Second Method.—Draw a line below the uppermost number, and suppose it cut off.—Then add all the rest of the numbers together in the usual way, and set their sum under the number to be proved.—Lastly, add this last found number and the uppermost line together; then if their sum be the same as that found by the first addition, it may be presumed the work is right.—This method of proof is founded on the plain axiom, that "The whole is equal to all its parts taken together."

Third Method.—Add the figures in the uppermost line together, and find how many nines are contained in their sum.—Reject those nines, and set down the remainder towards the right hand directly even with the figures in the line, as in the annexed example.—Do the same with each of the proposed lines of numbers, setting all these excesses of nines in a column on the right-hand, as here 5, 5, 6. Then, if the excess of 9's in this sum, found as before, be equal to the excess of 9's in the total sum 18304, the work is probably right.— Thus, the sum of the right-hand column, 5, 5, 6, is 16, the excess of which above 9 is 7. Also the sum of the figures in

EXAMPLE I.

	Excess of nines.	
3497		5
6512		5
8295		6
18304		7

the

the sum total 18304, is 16, the excess of which above 9 is also 7, the same as the former*.

OTHER EXAMPLES.

2.	3.	4.
12345	12345	12345
67890	67890	876
98765	9876	9087
43210	543	56
12345	21	234
67890	9	1012
302445	90684	23610
290100	78339	11265
302445	90684	23610

* This method of proof depends on a property of the number 9, which, except the number 3, belongs to no other digit whatever; namely, that "any number divided by 9, will leave the same remainder as the sum of its figures or digits divided by 9:" which may be demonstrated in this manner.

Demonstration. Let there be any number proposed, as 4658. This, separated into its several parts, becomes $4000 + 600 + 50 + 8$. But $4000 = 4 \times 1000 = 4 \times (999 + 1) = 4 \times 999 + 4$. In like manner $600 = 6 \times 99 + 6$; and $50 = 5 \times 9 + 5$. Therefore the given number $4658 = 4 \times 999 + 4 + 6 \times 99 + 6 + 5 \times 9 + 5 + 8 = 4 \times 999 + 6 \times 99 + 5 \times 9 + 4 + 6 + 5 + 8$; and $4658 \div 9 = (4 \times 999 + 6 \times 99 + 5 \times 9 + 4 + 6 + 5 + 8) \div 9$. But $4 \times 999 + 6 \times 99 + 5 \times 9$ is evidently divisible by 9, without a remainder; therefore if the given number 4658 be divided by 9, it will leave the same remainder as $4 + 6 + 5 + 8$ divided by 9. And the same, it is evident, will hold for any other number whatever.

In like manner, the same property may be shown to belong to the number 3; but the preference is usually given to the number 9, on account of its being more convenient in practice.

Now, from the demonstration above given, the reason of the rule itself is evident; for the excess of 9's in two or more numbers being taken separately, and the excess of 9's taken also out of the sum of the former excesses, it is plain that this last excess must be equal to the excess of 9's contained in the total sum of all these numbers; all the parts taken together being equal to the whole. ——This rule was first given by Dr. Wallis in his Arithmetic, published in the year 1657.

Ex-

Ex. 5. Add 3426; 9024; 5106; 8890; 1204, together. Ans. 27650.

6. Add 509267; 235809; 72920; 8392; 420; 21; and 9, together. Ans. 826838.

7. Add 2; 19; 817; 4298; 50916; 730205; 9180634, together. Ans. 9966891.

8. How many days are in the twelve calendar months? Ans. 365.

9. How many days are there from the 15th day of April to the 24th day of November, both days included? Ans. 224.

10. An army consisting of 52714 infantry*, or foot, 5110 horse, 6250 dragoons, 3927 light-horse, 928 artillery, or gunners, 1410 pioneers, 250 sappers, and 406 miners: what is the whole number of men? Ans. 70995.

OF SUBTRACTION.

Subtraction teaches to find how much one number exceeds another, called their *difference*, or the *remainder*, by taking the less from the greater. The method of doing which is as follows:

Place the less number under the greater, in the same manner as in Addition, that is, units under units, tens under tens, and so on; and draw a line below them.—Begin at the right hand, and take each figure in the lower line, or number, from the figure above it, setting down the remainder below it.—But if the figure in the lower line be greater than that above it, first borrow, or add, 10 to the upper one, and then take the lower figure from that sum, setting down the remainder, and carrying 1, for what was borrowed, to the next lower figure, with which proceed as before; and so on till the whole is finished.

* The whole body of foot soldiers is denoted by the word *Infantry;* and all those that charge on horseback by the word *Cavalry.*—Some authors conjecture that the term infantry is derived from a certain Infanta of Spain, who, finding that the army commanded by the king her father had been defeated by the Moors, assembled a body of the people together on foot, with which she engaged and totally routed the enemy. In honour of this event, and to distinguish the foot soldiers, who were not before held in much estimation, they received the name of Infantry, from her own title of Infanta.

To

To prove Subtraction.

Add the remainder to the less number, or that which is just above it; and if the sum be equal to the greater or uppermost number, the work is right*.

EXAMPLES.

1.	2.	3.
From 5386427	From 5386427	From 1234567
Take 2164315	Take 4258792	Take 702973
Rem. 3222112	Rem. 1127635	Rem. 531594
Proof. 5386427	Proof. 5386427	Proof. 1234567

4. From 5331806 take 5073918. Ans. 257888.
5. From 7020974 take 2766809. Ans. 4254165.
6. From 8503602 take 574271. Ans. 7929131.

7. Sir Isaac Newton was born in the year 1642, and he died in 1727: how old was he at the time of his decease? Ans. 85 years.

8. Homer was born 2543 years ago, and Christ 1810 years ago: then how long before Christ was the birth of Homer? Ans. 733 years.

9. Noah's flood happened about the year of the world 1656, and the birth of Christ about the year 4000: then how long was the flood before Christ? Ans. 2344 years.

10. The Arabian or Indian method of notation was first known in England about the year 1150: then how long is it since to this present year 1810? Ans. 660 years.

11. Gunpowder was invented in the year 1330: then how long was this before the invention of printing, which was in 1441? Ans. 111 years.

12. The mariner's compass was invented in Europe in the year 1302: then how long was that before the discovery of America by Columbus, which happened in 1492? Ans. 190 years.

* The reason of this method of proof is evident; for if the difference of two numbers be added to the less, it must manifestly make up a sum equal to the greater.

OF

OF MULTIPLICATION.

Multiplication is a compendious method of Addition, teaching how to find the amount of any given number when repeated a certain number of times; as, 4 times 6, which is 24.

The number to be multiplied, or repeated, is called the *Multiplicand*.—The number you multiply by, or the number of repetitions, is the *Multiplier*.—And the number found, being the total amount, is called the *Product*.—Also, both the multiplier and multiplicand are, in general, named the *Terms* or *Factors*.

Before proceeding to any operations in this rule, it is necessary to learn off very perfectly the following Table, of all the products of the first 12 numbers, commonly called the Multiplication Table, or sometimes Pythagoras's Table, from its inventor.

Multiplication Table.

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

To

To multiply any Given Number by a Single Figure, or by any Number not more than 12.

* Set the multiplier under the units figure, or right-hand place, of the multiplicand, and draw a line below it.—Then, beginning at the right-hand, multiply every figure in this by the multiplier.—Count how many tens there are in the product of every single figure, and set down the remainder directly under the figure that is multiplied; and if nothing remains, set down a cipher.—Carry as many units or ones as there are tens counted, to the product of the next figures; and proceed in the same manner till the whole is finished.

EXAMPLE.

Multiply	9876543210	the Multiplicand.
By - - - -	2	the Multiplier.
	19753086420	the Product.

To multiply by a Number consisting of Several Figures.

† Set the multiplier below the multiplicand, placing them as in Addition, namely, units under units, tens under tens, &c. drawing a line below it.—Multiply the whole of the multiplicand by each figure of the multiplier, as in the last article; setting

* The reason of this rule is the same as for the process in Addition, in which 1 is carried for every 10, to the next place, gradually as the several products are produced, one after another, instead of setting them all down one below each other, as in the annexed example.

5678		
4		
32	=	8 × 4
280	=	70 × 4
2400	=	600 × 4
20000	=	5000 × 4
22712	=	5678 × 4

† After having found the produce of the multiplicand by the first figure of the multiplier, as in the former case, the multiplier is supposed to be divided into parts, and the product is found for the second figure in the same manner: but as this figure stands in the place of tens, the product must be ten times its simple value; and therefore the first figure of this product must be set in the place of tens;

setting down a line of products for each figure in the multiplier, so as that the first figure of each line may stand straight under the figure multiplying by.—Add all the lines of products together, in the order as they stand, and their sum will be the answer or whole product required.

To prove Multiplication.

There are three different ways of proving Multiplication, which are as below:

First Method.—Make the multiplicand and multiplier change places, and multiply the latter by the former in the same manner as before. Then if the product found in this way be the same as the former, the number is right.

Second Method.—* Cast all the 9's out of the sum of the figures in each of the two factors, as in Addition, and set down the remainders. Multiply these two remainders together, and cast the 9's out of the product, as also out of

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tens; or, which is the same thing, directly under the figure multiplied by. And proceeding in this manner separately with all the figures of the multiplier, it is evident that we shall multiply all the parts of the multiplicand by all the parts of the multiplier, or the whole of the multiplicand by the whole of the multiplier: therefore these several products being added together, will be equal to the whole required product; as in the example annexed.

1234567		the multiplicand.
4567		
8641969	=	7 times the mult.
7407402	=	60 times ditto.
6172835	=	500 times ditto.
4938268	=	4000 times ditto.
5638267489	=	4567 times ditto.

* This method of proof is derived from the peculiar property of the number 9, mentioned in the proof of Addition, and the reason for the one may serve for that of the other. Another more ample demonstration of this rule may be as follows:—Let P and Q denote the number of 9's in the factors to be multiplied, and a and b what remain; then $9\,P+a$ and $9\,Q+b$ will be the numbers themselves, and their product is $(9\,P \times 9\,Q) + (9\,P \times b) + (9\,Q \times a) + (a \times b)$; but the first three of these products are each a precise number of 9's, because their factors are so, either one or both: these therefore being cast away, there remains only $a \times b$; and if the 9's also be cast out of this, the excess is the excess of 9's in the total product: but a and b are the excesses in the factors themselves, and $a \times b$ is their product; therefore the rule is true.

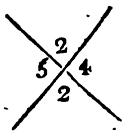
the

the whole product or answer of the question, reserving the remainders of these last two, which remainders must be equal when the work is right.—*Note*, It is common to set the four remainders within the four angular spaces of a cross, as in the example below.

Third Method.—Multiplication is also very naturally proved by Division; for the product divided by either of the factors, will evidently give the other. But this cannot be practised till the rule of Division is learned.

EXAMPLES.

Mult.	3542	Proof.	or Mult. 6196
by	6196		by 3542
	21252		12392
	31878		24784
	3542		30980
	21252		18588
	21946232 Product.		21946232 Proof.



OTHER EXAMPLES.

Multiply	123456789	by	3.	Ans.	370370367.
Multiply	123456789	by	4.	Ans.	493827156.
Multiply	123456789	by	5.	Ans.	617283945.
Multiply	123456789	by	6.	Ans.	740740734.
Multiply	123456789	by	7.	Ans.	864197523.
Multiply	123456789	by	8.	Ans.	987654312.
Multiply	123456789	by	9.	Ans.	1111111101.
Multiply	123456789	by	11.	Ans.	1358024679.
Multiply	123456789	by	12.	Ans.	1481481468.
Multiply	302914603	by	16.	Ans.	4846633648.
Multiply	273580961	by	23.	Ans.	6292362103.
Multiply	402097316	by	195.	Ans.	78408976620.
Multiply	82164973	by	3027.	Ans.	248713373271.
Multiply	7564900	by	579.	Ans.	4380077100.
Multiply	8496427	by	874359.	Ans.	7428927415293.
Multiply	2760325	by	37072.	Ans.	102330768400.

CONTRAC-

CONTRACTIONS IN MULTIPLICATION.

I. *When there are Ciphers in the Factors.*

If the ciphers be at the right-hand of the numbers; multiply the other figures only, and annex as many ciphers to the right-hand of the whole product, as are in both the factors.—When the ciphers are in the middle parts of the multiplier; neglect them as before, only taking care to place the first figure of every line of products exactly under the figure multiplying with.

EXAMPLES.

1.		2.
Mult. 9001635		Mult. 390720400
by 70100		by 406000
9001635		23443224
63011445		15628816
631014613500	Products	158632482400000

3. Multiply 81503600 by 7030. Ans. 572970308000.
4. Multiply 9030100 by 2100. Ans. 18963210000.
5. Multiply 8057069 by 70050. Ans. 564397683450.

II. *When the Multiplier is the Product of two or more Numbers in the Table; then*

* Multiply by each of those parts separately, instead of the whole number at once.

EXAMPLES.

1. Multiply 51307298 by 56, or 7 times 8.

```
 51307298
        7
---------
359151086
        8
---------
2873208688
```

* The reason of this rule is obvious enough; for any number multiplied by the component parts of another, must give the same product as if it were multiplied by that number at once. Thus, in the 1st example, 7 times the product of 8 by the given number, makes 56 times the same number, as plainly as 7 times 8 makes 56.

2. Multiply 31704592 by 36. Ans. 1141365312.
3. Multiply 29753804 by 72. Ans. 2142273888.
4. Multiply 7128368 by 96. Ans. 684323328.
5. Multiply 160430800 by 108. Ans. 17326526400.
6. Multiply 61835720 by 1320. Ans. 81623150400.

7. There was an army composed of 104 * battalions, each consisting of 500 men; what was the number of men contained in the whole? Ans. 52000.

8. A convoy of ammunition † bread, consisting of 250 waggons, and each waggon containing 320 loaves, having been intercepted and taken by the enemy; what is the number of loaves lost? Ans. 80000.

OF DIVISION.

Division is a kind of compendious method of Subtraction, teaching to find how often one number is contained in another, or may be taken out of it: which is the same thing.

The number to be divided is called the *Dividend.*—The number to divide by, is the *Divisor.*—And the number of times the dividend contains the divisor, is called the *Quotient.*—Sometimes there is a *Remainder* left, after the division is finished.

The usual manner of placing the terms, is, the dividend in the middle, having the divisor on the left hand, and the quotient on the right, each separated by a curve line; as, to divide 12 by 4, the quotient is 3,

Dividend
Divisor 4) 12 (3 Quotient;

showing that the number 4 is 3 times contained in 12, or may be 3 times subtracted out of it, as in the margin.

12
4 subtr.
—
8
4 subtr.
—
4
4 subtr.
—
0
—

‡ *Rule.*—Having placed the divisor before the dividend, as above directed, find how often the divisor is contained in as many figures of the dividend as are just necessary, and place the number on the right in the quotient. Mul-

* A battalion is a body of foot, consisting of 500, or 600, or 700 men, more or less.

† The ammunition bread, is that which is provided for, and distributed to, the soldiers; the usual allowance being a loaf of 6 pounds to every soldier, once in 4 days.

‡ In this way the dividend is resolved into parts, and by trial is found

Multiply the divisor by this number, and set the product under the figures of the dividend before-mentioned.—Subtract this product from that part of the dividend under which it stands, and bring down the next figure of the dividend, or more if necessary, to join on the right of the remainder.—Divide this number, so increased, in the same manner as before; and so on till all the figures are brought down and used.

N. B. If it be necessary to bring down more figures than one to any remainder, in order to make it as large as the divisor, or larger, a cipher must be set in the quotient for every figure so brought down more than one.

To prove Division.

* Multiply the quotient by the divisor; to this product add the remainder, if there be any; then the sum will be equal to the dividend when the work is right.

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found how often the divisor is contained in each of those parts, one after another, arranging the several figures of the quotient one after another, into one number.

When there is no remainder to a division, the quotient is the whole and perfect answer to the question. But when there is a remainder, it goes so much towards another time, as it approaches to the divisor: so, if the remainder be half the divisor, it will go the half of a time more; if the 4th part of the divisor, it will go one fourth of a time more; and so on. Therefore, to complete the quotient, set the remainder at the end of it, above a small line, and the divisor below it, thus forming a fractional part of the whole quotient.

* This method of proof is plain enough: for since the quotient is the number of times the dividend contains the divisor, the quotient multiplied by the divisor must evidently be equal to the dividend.

There are also several other methods sometimes used for proving Division, some of the most useful of which are as follow:

Second Method.—Subtract the remainder from the dividend, and divide what is left by the quotient; so shall the new quotient from this last division be equal to the former divisor, when the work is right.

Third Method.—Add together the remainder and all the products of the several quotient figures by the divisor, according to the order in which they stand in the work; and the sum will be equal to the dividend when the work is right.

EXAMPLES.

1.	Quot.
3) 1234567	(411522
12	mult. 3
3	1234566
3	add 1
4	1234567
3	
	Proof.
15	
15	
6	
6	
7	
6	
Rem. 1	

2.	Quot.
37) 12345678	(333666
111	37
124	2335662
111	1000998
	rem. 36
135	
111	12345678
246	Proof.
222	
247	
222	
258	
222	
Rem. 36	

3. Divide 73146085 by 4. Ans. $18286521\frac{1}{4}$.
4. Divide 5317986027 by 7. Ans. $759712289\frac{4}{7}$.
5. Divide 570196382 by 12. Ans. $47516365\frac{2}{12}$.
6. Divide 74638105 by 37. Ans. $2017246\frac{3}{37}$.
7. Divide 137896254 by 97. Ans. $1421610\frac{84}{97}$.
8. Divide 35821649 by 764. Ans. $46886\frac{745}{764}$.
9. Divide 72091365 by 5201. Ans. $13861\frac{304}{5201}$.
10. Divide 4637064283 by 57606. Ans. $80496\frac{11707}{57606}$.
11. Suppose 471 men are formed into ranks of 3 deep, what is the number in each rank? Ans. 157.
12. A party, at the distance of 378 miles from the head quarters, receive orders to join their corps in 18 days: what number of miles must they march each day to obey their orders? Ans. 21.
13. The annual revenue of a gentleman being 38330*l*; how much per day is that equivalent to, there being 365 days in the year? Ans. 104*l*.

Contractions in Division.

There are certain contractions in Division, by which the operation in particular cases may be performed in a shorter manner: as follows:

I. *Divi-*

I. *Division by any Small Number*, not greater than 12, may be expeditiously performed, by multiplying and subtracting mentally, omitting to set down the work, except only the quotient immediately below the dividend.

EXAMPLES.

3) 56103961   4) 52619675   5) 1379192

Quot. $18701320\frac{1}{3}$

6) 38672940   7) 81396627   8) 23718920

9) 43981962   11) 57614230   12) 27980373

II. * *When Ciphers are annexed to the Divisor;* cut off those ciphers from it, and cut off the same number of figures from the right-hand of the dividend; then divide with the remaining figures, as usual. And if there be any thing remaining after this division, place the figures cut off from the dividend to the right of it, and the whole will be the true remainder; otherwise, the figures cut off only will be the remainder.

EXAMPLES.

1. Divide 3704196 by 20.

2,0) 370419,6

Quot. $185209\frac{16}{20}$

2. Divide 31086901 by 7100.

71,00) 310869,01 ($4378\frac{3101}{7100}$.
284

268
213

556
497

599
568

31

3. Divide

* This method is only to avoid a needless repetition of ciphers, which would happen in the common way. And the truth of the principle

3. Divide 7380964 by 23000. Ans. $320\frac{20964}{23000}$.
4. Divide 2304109 by 5800. Ans. $397\frac{1509}{5800}$.

III. *When the Divisor is the exact Product of two or more of the small Numbers not greater than* 12: * Divide by each of those numbers separately, instead of the whole divisor at once.

N. B. There are commonly several remainders in working by this rule, one to each division; and to find the true or whole remainder, the same as if the division had been performed all at once, proceed as follows: Multiply the last remainder by the preceding divisor, or last but one, and to the product add the preceding remainder; multiply this sum by the next preceding divisor, and to the product add the next preceding remainder; and so on, till you have gone backward through all the divisors and remainders to the first. As in the example following:

EXAMPLES.

1. Divide 31046835 by 56 or 7 times 8.

7) 31046835	6 the last rem.
	mult. 7 preced. divisor.
8) 4435262—1 first rem.	42
554407—6 second rem.	add 1 the 1st rem.
Ans. $554407\frac{43}{56}$	43 whole rem.

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principle on which it is founded, is evident; for, cutting off the same number of ciphers, or figures, from each, is the same as dividing each of them by 10, or 100, or 1000, &c. according to the number of ciphers cut off; and it is evident, that as often as the whole divisor is contained in the whole dividend, so often must any part of the former be contained in a like part of the latter.

* This follows from the second contraction in Multiplication, being only the converse of it; for the half of the third part of any thing, is evidently the same as the sixth part of the whole; and so of any other numbers.—The reason of the method of finding the whole remainder from the several particular ones, will best appear from the nature of Vulgar Fractions. Thus, in the first example above, the first remainder being 1, when the divisor is 7, makes $\frac{1}{7}$; this must be added to the second remainder, 6, making $6\frac{1}{7}$ to the divisor 8, or to be divided by 8. But $6\frac{1}{7} = \frac{6 \times 7 + 1}{7} = \frac{43}{7}$; and this divided by 8 gives $\frac{43}{7 \times 8} = \frac{43}{56}$.

2. Divide

2. Divide 7014596 by 72. Ans. $97424\frac{68}{72}$.
3. Divide 5130652 by 132. Ans. $38868\frac{76}{132}$.
4. Divide 83016572 by 240. Ans. $345902\frac{92}{240}$.

IV. *Common Division may be performed more concisely*, by omitting the several products, and setting down only the remainders; namely, multiply the divisor by the quotient figures as before, and, without setting down the product, subtract each figure of it from the dividend, as it is produced; always remembering to carry as many to the next figure as were borrowed before.

EXAMPLES.

1. Divide 3104679 by 833.

```
833 ) 3104679 ( 3727 88/833.
        6056
         2257
          5919
            88
```

2. Divide 79165238 by 238. Ans. $332627\frac{12}{238}$.
3. Divide 29137062 by 5317. Ans. $5479\frac{5219}{5317}$.
4. Divide 62015735 by 7803. Ans. $7947\frac{5294}{7803}$.

OF REDUCTION.

REDUCTION is the changing of numbers from one name or denomination to another, without altering their value.—This is chiefly concerned in reducing money, weights, and measures.

When the numbers are to be reduced from a higher name to a lower, it is called *Reduction Descending*; but when, contrarywise, from a lower name to a higher, it is *Reduction Ascending*.

Before proceeding to the rules and questions of Reduction, it will be proper to set down the usual Tables of money, weights, and measures, which are as follow:

Of

Of MONEY, WEIGHTS, AND MEASURES.

TABLES OF MONEY*.

2 Farthings	=	1 Halfpenny	½	qrs		d				
4 Farthings	=	1 Penny	d	4	=	1		s		
12 Pence	=	1 Shilling	s	48	=	12	=	1		£
20 Shillings	=	1 Pound	£	960	=	240	=	20	=	1

PENCE TABLE.				SHILLINGS TABLE.		
d		s	d	s		d
20	is	1	8	1	is	12
30	—	2	6	2	—	24
40	—	3	4	3	—	36
50	—	4	2	4	—	48
60	—	5	0	5	—	60
70	—	5	10	6	—	72
80	—	6	8	7	—	84
90	—	7	6	8	—	96
100	—	8	4	9	—	108
110	—	9	2	10	—	120
120	—	10	0	11	—	132

TROY

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* £ denotes pounds, *s* shillings, and *d* denotes pence.
¼ denotes 1 farthing, or one quarter of any thing.
½ denotes a halfpenny, or the half of any thing.
¾ denotes 3 farthings, or three quarters of any thing.

The full weight and value of the English gold and silver coin, is as here below :

GOLD.	Value.			Weight.		SILVER.	Value.		Weight.	
	£	s	d	dwt	gr		s	d	dwt	gr
A Guinea	1	1	0	5	9½	A Crown	5	0	19	8½
Half-guinea	0	10	6	2	16¾	Half-crown	2	6	9	16¼
Seven Shillings	0	7	0	1	19⅙	Shilling	1	0	3	21
Quarter-guinea	0	5	3	1	8¼	Sixpence	0	6	1	22½

The usual value of gold is nearly *4l* an ounce, or *2d* a grain; and that of silver is nearly *5s* an ounce. Also, the value of any quantity of gold, is to the value of the same weight of standard silver, nearly as 15 to 1, or more nearly as 15 and 1-14th to 1.

Pure gold, free from mixture with other metals, usually called fine gold, is of so pure a nature, that it will endure the fire without

TROY WEIGHT*.

Grains	- -	marked *gr*	*gr*	*dwt*				
24 Grains make	1 Pennyweight	*dwt*	24=	1	*oz*			
20 Pennyweights	1 Ounce	*oz*	480=	20=	1	*lb*		
12 Ounces	1 Pound	*lb*	5760=	240=	12=	1		

By this weight are weighed Gold, Silver, and Jewels.

APOTHECARIES' WEIGHT.

Grains	- -	marked *gr*	
20 Grains make	1 Scruple	*sc* or	℈
3 Scruples	1 Dram	*dr* or	ʒ
8 Drams	1 Ounce	*oz* or	℥
12 Ounces	1 Pound	*lb* or	℔

gr		*sc*						
20	=	1		*dr*				
60	=	3	=	1		*oz*		
480	=	24	=	8	=	1		*lb*
5760	=	288	=	96	=	12	=	1

This is the same as Troy weight, only having some different divisions. Apothecaries make use of this weight in compounding their Medicines; but they buy and sell their Drugs by Avoirdupois weight.

AVOIR-

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without wasting, though it be kept continually melted. But silver, not having the purity of gold, will not endure the fire like it: yet fine silver will waste but a very little by being in the fire any moderate time; whereas copper, tin, lead, &c. will not only waste, but may be calcined, or burnt to a powder.

Both gold and silver, in their purity, are so very soft and flexible (like new lead, &c.), that they are not so useful, either in coin or otherwise (except to beat into leaf gold or silver), as when they are allayed, or mixed and hardened with copper or brass. And though most nations differ, more or less, in the quantity of such allay, as well as in the same place at different times, yet in England the standard for gold and silver coin has been for a long time as follows—viz. That 22 parts of fine gold, and 2 parts of copper, being melted together, shall be esteemed the true standard for gold coin: And that 11 ounces and 2 pennyweights of fine silver, and 18 pennyweights of copper, being melted together, is esteemed the true standard for silver coin, called Sterling silver.

* The original of all weights used in England was a grain or corn of wheat, gathered out of the middle of the ear, and, being well dried, 32 of them were to make one pennyweight, 20 penny-weights

Avoirdupois Weight.

	Drams	- - - -	marked	*dr*
16	Drams	make	1 Ounce - - -	*oz*
16	Ounces	- - -	1 Pound - - -	*lb*
28	Pounds	- - -	1 Quarter - - -	*qr*
4	Quarters	- - -	1 Hundred Weight -	*cwt*
20	Hundred Weight		1 Ton - - -	*ton*

dr		*oz*		*lb*		*qr*		*cwt*		*ton*
16	=	1								
256	=	16	=	1						
7168	=	448	=	28	=	1				
28672	=	1792	=	112	=	4	=	1		
573440	=	35840	=	2240	=	80	=	20	=	1

By this weight are weighed all things of a coarse or drossy nature, as Corn, Bread, Butter, Cheese, Flesh, Grocery Wares, and some Liquids; also all Metals, except Silver and Gold.

		oz	*dwt*	*gr*	
Note, that 1*lb* Avoirdupois	=	14	11	$15\frac{1}{2}$	Troy.
1*oz* - -	=	0	18	$5\frac{1}{2}$	
1*dr* - -	=	0	1	$3\frac{1}{2}$	

Long Measure.

3	Barley-corns	make	1 Inch - -	*In*
12	Inches	- -	1 Foot - -	*Ft*
3	Feet	- -	1 Yard - -	*Yd*
6	Feet	- -	1 Fathom - -	*Fth*
5	Yards and a half		1 Pole or Rod -	*Pl*
40	Poles	- -	1 Furlong - -	*Fur*
8	Furlongs	- -	1 Mile - -	*Mile*
3	Miles	- -	1 League - -	*Lea*
$69\frac{1}{2}$	Miles nearly	-	1 Degree - -	*Deg* or °.

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weights one ounce, and 12 ounces one pound. But in later times, it was thought sufficient to divide the same pennyweight into 24 equal parts, still called grains, being the least weight now in common use; and from thence the rest are computed, as in the Tables above.

In

In	*Ft*	*Yd*	*Pl*	*Fur*	*Mile*
12	= 1				
36	= 3	= 1			
198	= $16\frac{1}{2}$	= $5\frac{1}{2}$	= 1		
7920	= 660	= 220	= 40	= 1	
63360	= 5280	= 1760	= 320	= 8	= 1

Cloth Measure.

2 Inches and a quarter make	1	Nail	*Nl*
4 Nails	1	Quarter of a Yard	*Qr*
3 Quarters	1	Ell Flemish	*E F*
4 Quarters	1	Yard	*Yd*
5 Quarters	1	Ell English	*E E*
4 Quarters $1\frac{1}{5}$ Inch	1	Ell Scotch	*E S*

Square Measure.

144	Square Inches make	1	Sq Foot	*Ft*
9	Square Feet	1	Sq Yard	*Yd*
$30\frac{1}{4}$	Square Yards	1	Sq Pole	*Pole*
40	Square Poles	1	Rood	*Rd*
4	Roods	1	Acre	*Acr*

Sq Inc	*Sq Ft*	*Sq Yd*	*Sq Pl*	*Rd*	*Acr*
144	= 1				
1296	= 9	= 1			
39204	= $272\frac{1}{4}$	= $30\frac{1}{4}$	= 1		
1568160	= 10890	= 1210	= 40	= 1	
6272640	= 43560	= 4840	= 160	= 4	= 1

By this measure, Land, and Husbandmen and Gardeners' work are measured; also Artificers' work, such as Board, Glass, Pavements, Plastering, Wainscoting, Tiling, Flooring, and every dimension of length and breadth only.

When three dimensions are concerned, namely, length, breadth, and depth or thickness, it is called cubic or solid measure, which is used to measure Timber, Stone, &c.

The cubic or solid Foot, which is 12 inches in length and breadth and thickness, contains 1728 cubic or solid inches, and 27 solid feet make one solid yard.

DRY,

DRY, *or* CORN MEASURE.

2	Pints make	1	Quart	- -	*Qt*
2	Quarts -	1	Pottle	- -	*Pot*
2	Pottles -	1	Gallon	- -	*Gal*
2	Gallons -	1	Peck	- -	*Pec*
4	Pecks -	1	Bushel	- -	*Bu*
8	Bushels -	1	Quarter	- -	*Qr*
5	Quarters -	1	Wey, Load, or Ton		*Wey*
2	Weys -	1	Last	- - -	*Last*

Pts		*Gal*										
8	=	1		*Pec*								
16	=	2	=	1		*Bu*						
64	=	8	=	4	=	1		*Qr*				
512	=	64	=	32	=	8	=	1		*Wey*		
2560	=	320	=	160	=	40	=	5	=	1		*Last*
5120	=	640	=	320	=	80	=	10	=	2	=	1

By this are measured all dry wares, as, Corn, Seeds, Roots, Fruits, Salt, Coals, Sand, Oysters, &c.

The standard Gallon dry-measure contains $268\frac{4}{5}$ cubic or solid inches, and the Corn or Winchester bushel $2150\frac{2}{5}$ cubic inches; for the dimensions of the Winchester bushel, by the Statute, are 8 inches deep, and $18\frac{1}{2}$ inches wide or in diameter. But the Coal bushel must be $19\frac{1}{2}$ inches in diameter; and 36 bushels, heaped up, make a London chaldron of coals, the weight of which is 3156lb Avoirdupois.

ALE *and* BEER MEASURE.

2	Pints make -	1	Quart	-	*Qt*
4	Quarts - -	1	Gallon	-	*Gal*
36	Gallons - -	1	Barrel	-	*Bar*
1	Barrel and a half	1	Hogshead		*Hhd*
2	Barrels - -	1	Puncheon		*Pun*
2	Hogsheads -	1	Butt	-	*Butt*
2	Butts - -	1	Tun	-	*Tun*

Pts		*Qt*								
2	=	1		*Gal*						
8	=	4	=	1		*Bar*				
288	=	144	=	36	=	1		*Hhd*		
432	=	216	=	54	=	$1\frac{1}{2}$	=	1		*Butt*
864	=	432	=	108	=	3	=	2	=	1

Note, The Ale Gallon contains 282 cubic or solid Inches.

WINE

WINE MEASURE.

2 Pints make	1 Quart	-	*Qt*
4 Quarts	1 Gallon	-	*Gal*
42 Gallons	1 Tierce	-	*Tier*
63 Gallons or $1\frac{1}{2}$ Tierces	1 Hogshead	-	*Hhd*
2 Tierces	1 Puncheon	-	*Pun*
2 Hogsheads	1 Pipe or Butt		*Pi*
2 Pipes or 4 Hhds	1 Tun	-	*Tun*

Pts		*Qt*												
2	=	1		*Gal*										
8	=	4	=	1		*Tier*								
336	=	168	=	42	=	1		*Hhd*						
504	=	252	=	63	=	$1\frac{1}{2}$	=	1		*Pun*				
672	=	336	=	84	=	2	=	$1\frac{1}{2}$	=	1		*Pi*		
1008	=	504	=	126	=	3	=	2	=	$1\frac{1}{2}$	=	1		*Tun*
2016	=	1008	=	252	=	6	=	4	=	3	=	2	=	1

Note, By this are measured all Wines, Spirits, Strong-waters, Cyder, Mead, Perry, Vinegar, Oil, Honey, &c.

The Wine Gallon contains 231 cubic or solid inches. And it is remarkable, that the Wine and Ale Gallons have the same proportion to each other, as the Troy and Avoirdupois Pounds have; that is, as one Pound Troy is to one Pound Avoirdupois, so is one Wine Gallon to one Ale Gallon.

Of TIME.

60 Seconds or 60″ make	1 Minute	-	*M* or ′
60 Minutes	1 Hour	-	*Hr*
24 Hours	1 Day	-	*Day*
7 Days	1 Week	-	*Wk*
4 Weeks	1 Month	-	*Mo*
13 Months 1 Day 6 Hours, or 365 Days 6 Hours	1 Julian Year		*Yr*

Sec		*Min*										
60	=	1		*Hr*								
3600	=	60	=	1		*Day*						
86400	=	1440	=	24	=	1		*Wk*				
604800	=	10080	=	168	=	7	=	1		*Mo*		
2419200	=	40320	=	672	=	28	=	4	=	1		
31557600	=	525960	=	8766	=	$365\frac{1}{4}$	=					1 *Year*,

Or

	Wk	Da	Hr		Mo	Da	Hr		
Or	52	1	6	=	13	1	6	=	1 *Julian Year*

	Da	Hr	M	Sec		
But	365	5	48	48	=	1 *Solar Year.*

RULES FOR REDUCTION.

I. *When the Numbers are to be reduced from a Higher Denomination to a Lower:*

MULTIPLY the number in the highest denomination by as many as of the next lower make an integer, or 1, in that higher; to this product add the number, if any, which was in this lower denomination before, and set down the amount.

Reduce this amount in like manner, by multiplying it by as many as of the next lower make an integer of this, taking in the odd parts of this lower, as before. And so proceed through all the denominations to the lowest; so shall the number last found be the value of all the numbers which were in the higher denominations, taken together *.

EXAMPLE.

1. In 1234*l* 15*s* 7*d*, how many farthings?

	l	*s*	*d*
	1234	15	7
	20		
	24695	Shillings	
	12		
	296347	Pence	
	4		
Answer	1185388	Farthings.	

* The reason of this rule is very evident; for pounds are brought into shillings by multiplying them by 20; shillings into pence, by multiplying them by 12; and pence into farthings, by multiplying by 4; and the reverse of this rule by Division.—And the same, it is evident, will be true in the reduction of numbers consisting of any denominations whatever.

II. *When*

II. *When the Numbers are to be reduced from a Lower Denomination to a Higher:*

DIVIDE the given number by as many as of that denomination make 1 of the next higher, and set down what remains, as well as the quotient.

Divide the quotient by as many as of this denomination make 1 of the next higher; setting down the new quotient, and remainder, as before.

Proceed in the same manner through all the denominations, to the highest; and the quotient last found, together with the several remainders, if any, will be of the same value as the first number proposed.

EXAMPLES.

2. Reduce 1185388 farthings into pounds, shillings, and pence.

```
       4) 1185388
          -------
      12)  296347 d
          -------
     2,0)  2469,5 s—7d
          -------
Answer    1234l 15s 7d
          ----------
```

3. Reduce 24*l* to farthings. Ans. 23040.
4. Reduce 337587 farthings to pounds, &c.
Ans. 351*l* 13*s* 0¾.
5. How many farthings are in 36 guineas? Ans. 36288.
6. In 36288 farthings how many guineas? Ans. 36.
7. In 59 lb 13 dwts 5 gr how many grains? Ans. 340157.
8. In 8012131 grains how many pounds, &c.?
Ans. 1390 lb 11 oz 18 dwt 19 gr
9. In 35 ton 17 cwt 1 qr 23 lb 7 oz 13 dr how many drams?
Ans. 20571005.

10. How many barley-corns will reach round the earth, supposing it, according to the best calculations, to be 25000 miles? Ans. 4752000000.

11. How many seconds are in a solar year, or 365 days 5 hrs 48 min 48 sec? Ans. 31556928.

12. In a lunar month, or 29 ds 12 hrs 44 min 3 sec, how many seconds? Ans. 2551443.

COM-

COMPOUND ADDITION.

Compound Addition shows how to add or collect several numbers of different denominations into one sum.

Rule.—Place the numbers so, that those of the same denomination may stand directly under each other, and draw a line below them. Add up the figures in the lowest denomination, and find, by Reduction, how many units, or ones, of the next higher denomination are contained in their sum.—Set down the remainder below its proper column, and carry those units or ones to the next denomination, which add up in the same manner as before.—Proceed thus through all the denominations, to the highest, whose sum, together with the several remainders, will give the answer sought.

The method of proof is the same as in Simple Addition.

EXAMPLES OF MONEY.

1.			2.			3.			4.		
l	*s*	*d*	*l*	*s*	*d*	*l*	*s*	*d*	*l*	*s*	*d*
7	13	3	14	7	5	15	17	10	53	14	8
3	5	10½	8	19	2¼	3	14	6	5	10	2¾
6	18	7	7	8	1½	23	6	2¾	93	11	6
0	2	5¾	21	2	9	14	9	4½	7	5	0
4	0	3	7	16	8½	15	6	4	13	2	5
17	15	4½	0	4	3	6	12	9¾	0	18	7
39	15	9¾									
32	2	6¾									
39	15	9¾									

5.			6.			7.			8.		
l	*s*	*d*	*l*	*s*	*d*	*l*	*s*	*d*	*l*	*s*	*d*
14	0	7¼	37	15	8	61	3	2½	472	15	3
8	15	3	14	12	9¾	7	16	8	9	2	2½
62	4	7	17	14	9	29	13	10¾	27	12	6¼
4	17	8	23	10	9¼	12	16	2	370	16	2½
23	0	4¾	8	6	0	0	7	5¼	13	7	4
6	6	7	14	0	5½	24	13	0	6	10	5¼
91	0	10¼	54	2	7½	5	0	10¾	30	0	11¾

EXAM-

EXAM. 9. A nobleman, going out of town, is informed by his steward, that his butcher's bill comes to 197*l* 13*s* 7½*d*; his baker's to 59*l* 5*s* 2¾*d*; his brewer's to 85*l*; his wine-merchant's to 103*l* 13*s*; to his corn-chandler is due 75*l* 3*d*; to his tallow-chandler and cheesemonger, 27*l* 15*s* 11¼*d*; and to his tailor 55*l* 3*s* 5¾*d*; also for rent, servants' wages, and other charges, 127*l* 3*s*: Now, supposing he would take 100*l* with him, to defray his charges on the road, for what sum must he send to his banker? Ans. 830*l* 14*s* 6¼*d*.

10. The strength of a regiment of foot, of 10 companies, and the amount of their subsistence*, for a month of 30 days, according to the annexed Table, are required?

Numb.	Rank.	Subsistence for a Month. *l*	*s*	*d*
1	Colonel	27	0	0
1	Lieutenant Colonel	19	10	0
1	Major	17	5	0
7	Captains	78	15	0
11	Lieutenants	57	15	0
9	Ensigns	40	10	0
1	Chaplain	7	10	0
1	Adjutant	4	10	0
1	Quarter-Master	5	5	0
1	Surgeon	4	10	0
1	Surgeon's Mate	4	10	0
30	Serjeants	45	0	0
30	Corporals	30	0	0
20	Drummers	20	0	0
2	Fifes	2	0	0
390	Private Men	292	10	0
507	Total	656	10	0

* Subsistence Money, is the money paid to the soldiers weekly which is short of their full pay, because their clothes, accoutrements, &c. are to be accounted for. It is likewise the money advanced to officers till their accounts are made up, which is commonly once a year, when they are paid their arrears. The following Table shows the full pay and subsistence of each rank on the English establishment.

DAILY PAY OF COMMISSIONED OFFICERS.

RANK.	Horse Artillery & corps of C.Cm.	D. G. Dr. & F. Cav.	Foot Artillery.	Regular and Fenc. Inf. and Militia.	Life Guards.		Horse Guards.		Foot Guards.	
					Subsist.	FullPay.	Subsist.	FullPay.	Subsist.	Full Pay.
Colonel (Comm) . .	—	1 12 10	2 4 0	1 2 6	1 7 0	1 10 0	1 11 0	—	1 10 0	1 19 0
Colonel (en Second)	1 10 0	—	1 4 0							
1st Lieut. Col.	1 6 0	1 3 0	1 0 0	0 15 11	1 3 3	1 11 0	1 2 6	—	1 1 6	1 8 6
2d Ditto	—	—	0 17 0							
1st Major	1 1 0	0 19 3	0 15 0	0 14 1	0 19 6	1 6 0	1 1 6	—	0 18 6	1 4 6
2d Ditto	—	—	—	—	0 18 0	1 4 0				
Captain	0 15 0	0 14 7	0 10 0	0 9 5	0 12 1	0 16 0	0 16 6	—	0 12 6	0 16 6
Capt. Lieut.	0 10 0	0 9 0	0 7 0	0 5 8	0 8 2	0 11 0	0 11 6	0 15 0	0 6 0	0 7 10
1st Lieut.	0 9 0	0 9 0	0 6 0	0 5 8						
2d Ditto	0 8 0	—	0 5 0							
Cornet	—	0 8 0	—	—	—	—	0 11 0	0 14 0		
Ensign	—	—	—	0 4 8	—	—	—	—	0 4 6	0 5 10
Adjutant	—	0 5 0	—	0 5 0	0 8 6	0 11 0	0 4 6	0 5 0	0 3 0	0 4 0
Pay-master										
Quarter-master . . .	—	0 5 0	—	0 5 8	—	—	0 6 6	0 8 6	—	0 5 8
Surgeon major . . .	0 12 0	0 11 4	0 10 0	0 9 5	—	—	—	—	0 12 6	0 15 0
Bat. Surg. or Surg.	—	—	—	—	0 6 0	0 8 0	0 9 0	0 12 0	0 7 6	0 10 0
Assist. Surg.	0 6 0	0 5 0	0 5 0	0 5 0	—	—	0 5 0	0 5 0	0 5 0	0 5 0
Veter Surg.	0 8 0	0 8 0								
Solicitor										

N. B. When a *Lieutenant*, *Ensign*, *Adjutant*, or *Quarter-master* of Foot, Militia, Fencible Infantry, or Invalids, holds two commissions, one shilling per day is to be deducted from the above rates for each commission.

EXAMPLES

EXAMPLES OF WEIGHTS, MEASURES, &c.

TROY WEIGHT.

1.			2.		
lb	oz	dwt	oz	dwt	gr
17	3	15	37	9	3
7	9	4	9	5	3
0	10	7	8	12	12
9	5	0	17	7	8
176	2	17	5	9	0
23	11	12	3	0	19

APOTHECARIES' WEIGHT.

3.				4.			
lb	oz	dr	sc	oz	dr	sc	gr
3	5	7	2	3	5	1	17
13	7	3	0	7	3	2	5
19	10	6	2	16	7	0	12
0	9	1	2	7	3	2	9
36	3	5	0	4	1	2	18
5	8	6	1	36	4	1	14

AVOIRDUPOIS WEIGHT.

5.			6.		
lb	oz	dr	cwt	qr	lb
17	10	13	15	2	15
5	14	8	6	3	24
12	9	18	9	1	14
27	1	6	9	1	17
0	4	0	10	2	6
6	14	10	3	0	3

LONG MEASURE.

7.			8.		
mls	fur	pls	yds	feet	inc
29	3	14	127	1	5
19	6	29	12	2	9
7	0	24	10	0	10
9	1	37	54	1	11
7	0	3	5	2	7
4	5	9	23	0	5

CLOTH MEASURE.

9.			10.		
yds	qr	nls	elen	qrs	nls
26	3	1	270	1	0
13	1	2	57	4	3
9	1	2	18	1	2
217	0	3	0	3	2
9	1	0	10	1	0
55	3	1	4	4	1

LAND MEASURE.

11.			12.		
ac	ro	p	ac	ro	p
225	3	37	19	0	16
16	1	25	270	3	29
7	2	18	6	3	13
4	2	9	23	0	34
42	1	19	7	2	16
7	0	6	75	0	23

WINE MEASURE.

13.			14.		
t	hds	gal	hds	gal	pts
13	3	15	15	61	5
8	1	37	17	14	13
14	1	20	29	23	7
25	0	12	3	15	1
3	1	9	16	8	0
72	3	21	4	36	6

ALE and BEER MEASURE.

15.			16.		
hds	gal	pts	hds	gal	pts
17	37	3	29	43	5
9	10	15	12	19	7
3	6	2	14	16	6
5	14	0	6	8	1
12	9	6	57	13	4
8	42	4	5	6	0

COMPOUND SUBTRACTION.

COMPOUND SUBTRACTION shows how to find the difference between any two numbers of different denominations. To perform which, observe the following Rule:

* PLACE the less number below the greater, so that the parts of the same denomination may stand directly under each other; and draw a line below them.—Begin at the right-hand, and subtract each number or part in the lower line, from the one just above it, and set the remainder straight below it.—But if any number in the lower line be greater than that above it, add as many to the upper number as make 1 of the next higher denomination; then take the lower number from the upper one thus increased, and set down the remainder. Carry the unit borrowed to the next number in the lower line; after which subtract this number from the one above it, as before; and so proceed till the whole is finished. Then the several remainders, taken together, will be the whole difference sought.

The method of proof is the same as in Simple Subtraction.

EXAMPLES OF MONEY.

	1.			2.			3.			4.		
	l	*s*	*d*	*l*	*s*	*d*	*l*	*s*	*d*	*l*	*s*	*d*
From	79	17	8¾	103	3	2½	81	10	11	254	12	0
Take	35	12	4¼	71	12	5¾	29	13	3¼	37	9	4¾
Rem.	44	5	4½	31	10	8¾						
Proof	79	17	8¾	103	3	2½						

5. What is the difference between 73*l* 5¼*d* and 19*l* 13*s* 10*d*?
Ans. 53*l* 6*s* 7¼*d*.

---

* The reason of this Rule will easily appear from what has been said in Simple Subtraction; for the borrowing depends on the same principle, and is only different as the numbers to be subtracted are of different denominations.

Ex. 6.

Ex 6. A lends to B 100*l*, how much is B in debt after A has taken goods of him to the amount of 73*l* 12*s* $4\frac{3}{4}$*d*?

Ans. 26*l* 7*s* $7\frac{1}{4}$*d*.

7. Suppose that my rent for half a year is 20*l* 12*s*, and that I have laid out for the land-tax 14*s* 6*d*, and for several repairs 1*l* 3*s* $3\frac{1}{4}$*d*, what have I to pay of my half-year's rent?

Ans. 18*l* 14*s* $2\frac{3}{4}$*d*.

8. A trader, failing, owes to A 35*l* 7*s* 6*d*, to B 91*l* 13*s* $\frac{1}{2}$*d*, to C 53*l* $7\frac{1}{4}$*d*, to D 87*l* 5*s*, and to E 111*l* 3*s* $5\frac{3}{4}$*d*. When this happened, he had by him in cash 23*l* 7*s* 5*d*, in wares 53*l* 11*s* $10\frac{1}{4}$*d*, in household furniture 63*l* 17*s* $7\frac{3}{4}$*d*, and in recoverable book-debts 25*l* 7*s* 5*d*. What will his creditors lose by him, suppose these things delivered to them?

Ans. 212*l* 5*s* $3\frac{1}{2}$*d*.

EXAMPLES OF WEIGHTS, MEASURES, &c.

TROY WEIGHT. — APOTHECARIES WEIGHT.

	1.				2.				3.				
	lb	oz	dwt	gr	lb	oz	dwt	gr	lb	oz	dr	scr	gr
From	9	2	12	10	7	10	4	17	73	4	7	0	14
Take	5	4	6	17	3	7	16	12	29	5	3	4	19
Rem.													
Proof													

AVOIRDUPOIS WEIGHT. — LONG MEASURE.

	4.			5.			6.			7.		
	c	qrs	lb	lb	oz	dr	m	fu	pl	yd	ft	in
From	5	0	17	71	5	9	14	3	17	96	0	4
Take	2	3	10	17	9	18	7	6	11	72	2	9
Rem.												
Proof												

CLOTH MEASURE. — LAND MEASURE.

	8.			9.			10.			11.		
	yd	qr	nl	yd	qr	nl	ac	ro	p	ac	ro	p
From	17	2	1	9	0	2	17	1	14	57	1	16
Take	9	0	2	7	2	1	16	2	8	22	3	29
Rem.												
Proof												

WINE

WINE MEASURE.

	12.			13.		
	t	hd	gal	hd	gal	pt
From	17	2	23	5	0	4
Take	9	1	36	2	12	6
Rem.						
Proof						

ALE and BEER MEASURE.

	14.			15.		
	hd	gal	pt	hd	gal	pt
From	14	29	3	71	16	5
Take	9	35	7	19	7	1
Rem.						
Proof						

DRY MEASURE.

	16.			17.		
	la	qr	bu	bu	gal	pt
From	9	4	7	13	7	1
Take	6	3	5	9	2	7
Rem.						
Proof						

TIME.

	18.			19.		
	mo	we	da	ds	hrs	min
From	71	2	5	114	17	26
Take	17	1	6	72	10	37
Rem.						
Proof						

20. The line of defence in a certain polygon being 236 yards, and that part of it which is terminated by the curtain and shoulder being 146 yards 1 foot 4 inches; what then was the length of the face of the bastion? Ans. 89 yds 1 ft 8 in.

COMPOUND MULTIPLICATION.

Compound Multiplication shows how to find the amount of any given number of different denominations repeated a certain proposed number of times; which is performed by the following rule.

Set the multiplier under the lowest denomination of the multiplicand, and draw a line below it.—Multiply the number in the lowest denomination by the multiplier, and find how many units of the next higher denomination are contained in the product, setting down what remains.—In like manner, multiply the number in the next denomination, and to the product carry or add the units, before found, and find how many units of the next higher denomination are in this amount,

amount, which carry in like manner to the next product, setting down the overplus.—Proceed thus to the highest denomination proposed: so shall the last product, with the several remainders, taken as one compound number, be the whole amount required.—The method of Proof, and the reason of the Rule, are the same as in Simple Multiplication.

EXAMPLES OF MONEY.

1. To find the amount of 8 lb of Tea, at 5s $8\frac{1}{2}d$ per lb.

	s	d	
	5	$8\frac{1}{2}$	
		8	
£2	5	8	Answer.

		l	s	d
2.	4 lb of Tea, at 7s 8d per lb. Ans.	1	10	8
3.	6 lb of Butter, at $9\frac{1}{2}d$ per lb. Ans.	0	4	9
4.	7 lb of Tobacco, at 1s $8\frac{1}{2}d$ per lb. Ans.	0	11	$11\frac{1}{2}$
5.	9 stone of Beef, at 2s $7\frac{1}{4}d$ per st. Ans.	1	1	0
6.	10 cwt of Cheese, at 2l 17s 10d per cwt. Ans.	28	18	4
7.	12 cwt of Sugar, at 3l 7s 4d per cwt. Ans.	40	8	0

CONTRACTIONS.

I. If the multiplier exceed 12, multiply successively by its component parts, instead of the whole number at once.

EXAMPLES.

1. 15 cwt of Cheese, at 17s 6d per cwt.

l	s	d	
0	17	6	
		3	
2	12	6	
		5	
13	2	6	Answer.

		l	s	d
2.	20 cwt of Hops, at 4l 7s 2d per cwt. Ans.	87	3	4
3.	24 tons of Hay, at 3l 7s 6d per ton. Ans.	81	0	0
4.	45 ells of cloth, at 1s 6d per ell. Ans.	3	7	6

Ex. 5.

		l	s	d
Ex. 5. 63 gallons of Oil, at 2s 3d per gall.	Ans.	7	1	9
6. 70 barrels of Ale, at 1l 4s per barrel.	Ans.	84	0	0
7. 84 quarters of Oats, at 1l 12s 8d per qr.	Ans.	137	4	0
8. 96 quarters of Barley, at 1l 3s 4d per qr.	Ans.	112	0	0
9. 120 days' Wages, at 5s 9d per day.	Ans.	34	10	0
10. 144 reams of Paper, at 13s 4d per ream.	Ans.	96	0	0

II. If the multiplier cannot be exactly produced by the multiplication of simple numbers, take the nearest number to it, either greater or less, which can be so produced, and multiply by its parts, as before.—Then multiply the given multiplicand by the difference between this assumed number and the multiplier, and add the product to that before found, when the assumed number is less than the multiplier, but subtract the same when it is greater.

EXAMPLES.

1. 26 yards of Cloth, at 3s $0\frac{1}{4}d$ per yard.

l	s	d	
0	3	$0\frac{1}{4}$	
		5	
0	15	$3\frac{1}{4}$	
		5	
3	16	$6\frac{1}{4}$	
	3	$0\frac{1}{4}$	
£3	19	$7\frac{1}{2}$	Answer.

		l	s	d
2. 29 quarters of Corn, at 2l 5s $3\frac{1}{4}d$ per qr.	Ans.	65	12	$10\frac{1}{4}$
3. 53 loads of Hay, at 3l 15s 2d per load.	Ans.	199	3	10
4. 79 bushels of Wheat, at 11s $5\frac{3}{4}d$ per bush.	Ans.	45	6	$10\frac{1}{4}$
5. 97 casks of Beer, at 12s 2d per cask.	Ans.	59	0	2
6. 114 stone of Meat, at 15s $3\frac{3}{4}d$ per stone.	Ans.	87	5	$7\frac{1}{2}$

EXAMPLES OF WEIGHTS AND MEASURES.

1.

lb	oz	dwt	gr
28	7	14	10
			5

2.

lb	oz	dr	sc	gr
2	6	3	2	10
				8

3.

cwt	qr	lb	oz
29	2	16	14
			12

4.			
mls	fu	pls	yds
22	5	29	6
			4

5.		
yds	qrs	na
126	3	1
		7

6.		
ac	ro	po
28	3	27
		9

7.			
tuns	hhd	gal	pts
20	2	26	2
			3

8.			
we	qr	bu	pe
24	2	5	3
			6

9.				
mo	we	da	ho	min
172	3	5	16	49
				10

COMPOUND DIVISION.

Compound Division teaches how to divide a number of several denominations by any given number, or into any number of equal parts; as follows:

Place the divisor on the left of the dividend, as in Simple Division.—Begin at the left-hand, and divide the number of the highest denomination by the divisor, setting down the quotient in its proper place.—If there be any remainder after this division, reduce it to the next lower denomination, which add to the number, if any, belonging to that denomination, and divide the sum by the divisor.—Set down again this quotient, reduce its remainder to the next lower denomination again, and so on through all the denominations to the last.

EXAMPLES OF MONEY.

1. Divide 237*l* 8*s* 6*d* by 2.

	l	*s*	*d*	
2)	237	8	6	
£	118	14	3	the Quotient.

2. Divide

		l	s	d			l	s	d
2.	Divide	432	12	$1\frac{3}{4}$	by 3.	Ans.	144	4	$0\frac{1}{4}$
3.	Divide	507	3	5	by 4.	Ans.	126	15	$10\frac{1}{4}$
4.	Divide	632	7	$6\frac{1}{2}$	by 5.	Ans.	126	9	6
5.	Divide	690	14	$3\frac{1}{4}$	by 6.	Ans.	115	2	$4\frac{1}{2}$
6.	Divide	705	10	2	by 7.	Ans.	100	15	$8\frac{3}{4}$
7.	Divide	760	5	6	by 8.	Ans.	95	0	$8\frac{1}{4}$
8.	Divide	761	5	$7\frac{3}{4}$	by 9.	Ans.	84	11	$8\frac{3}{4}$
9.	Divide	829	17	10	by 10.	Ans.	82	19	$9\frac{1}{4}$
10.	Divide	937	8	$8\frac{3}{4}$	by 11.	Ans.	85	4	5
11.	Divide	1145	11	$4\frac{1}{4}$	by 12.	Ans.	95	9	$3\frac{1}{4}$

CONTRACTIONS.

I. If the divisor exceed 12, find what simple numbers, multiplied together, will produce it, and divide by them separately, as in Simple Division, as below.

EXAMPLES.

I. What is Cheese per cwt, if 16 cwt cost 25*l* 14*s* 8*d*?

	l	s	d
4)	25	14	8
4)	6	8	8
£	1	12	2 the Answer.

		l	s	d
2. If 20 cwt of Tobacco come to 150*l* 6*s* 8*d*, what is that per cwt?	Ans.	7	10	4
3. Divide 98*l* 8*s* by 36.	Ans.	2	14	8
4. Divide 71*l* 13*s* 10*d* by 56.	Ans.	1	5	$7\frac{1}{4}$
5. Divide 44*l* 4*s* by 96.	Ans.	0	9	$2\frac{1}{2}$
6. At 31*l* 10*s* per cwt, how much per lb?	Ans.	0	5	$7\frac{1}{2}$

II. If the divisor cannot be produced by the multiplication of small numbers, divide by the whole divisor at once, after the manner of Long Division, as follows.

EXAM-

EXAMPLES.

1. Divide 59*l* 6*s* $3\frac{3}{4}$*d* by 19.

	l	*s*	*d*	*l*	*s*	*d*	
19)	59	6	$3\frac{3}{4}$	(3	2	$5\frac{1}{4}$	Ans.
	57						
	2						
	20						
	46	(2					
	38						
	8						
	12						
	99	(5					
	95						
	4						
	4						
	19	(1					

	l	*s*	*d*				*l*	*s*	*d*
2. Divide	39	14	$5\frac{1}{4}$	by	57.	Ans.	0	13	$11\frac{1}{4}$
3. Divide	125	4	9	by	43.	Ans.	2	18	3
4. Divide	542	7	10	by	97.	Ans.	5	11	10
5. Divide	123	11	$2\frac{1}{2}$	by	127.	Ans.	0	19	$5\frac{1}{4}$

EXAMPLES OF WEIGHTS AND MEASURES.

1. Divide 17 lb 9 oz 0 dwts 2 gr by 7.
   Ans. 2 lb 6 oz 8 dwts 14 gr.
2. Divide 17 lb 5 oz 2 dr 1 scr 4 gr by 12.
   Ans. 1 lb 5 oz 3 dr 1 scr 12 gr.
3. Divide 178 cwt 3 qrs 14 lb by 53. Ans. 3 cwt 1 qr 14 lb.
4. Divide 144 mi 4 fur 2 po 1 yd 2 ft 0 in by 39.
   Ans. 3 mi 5 fur 26 po 0 yds 2 ft 8 in.
5. Divide 534 yds 2 qrs 2 na by 47. Ans. 11 yds 1 qr 2 na.
6. Divide 71 ac 1 ro 33 po by 51. Ans. 1 ac 2 ro 3 po.
7. Divide 7 tu 0 hhds 47 gal 7 pi by 65. Ans. 27 gal. 7 pi.
8. Divide 387 la 9 qr by 72. Ans. 5 la 3 qrs 7 bu.
9. Divide 206 mo 4 da by 26. Ans. 7 mo 3 we 5 ds.

THE

The GOLDEN RULE, or RULE OF THREE.

The Rule of Three teaches how to find a fourth proportional to three numbers given: for which reason it is sometimes called the Rule of Proportion. It is called the Rule of Three, because three terms or numbers are given, to find a fourth. And because of its great and extensive usefulness, it is often called the Golden Rule. This Rule is usually considered as of two kinds, namely, Direct, and Inverse.

The Rule of Three Direct is that in which more requires more, or less requires less. As in this; if 3 men dig 21 yards of trench in a certain time, how much will 6 men dig in the same time? Here more requires more, that is, 6 men, which are more than 3 men, will also perform more work, in the same time. Or when it is thus: if 6 men dig 42 yards, how much will 3 men dig in the same time? Here then, less requires less, or 3 men will perform proportionably less work than 6 men, in the same time. In both these cases then, the Rule, or the Proportion, is Direct; and the stating must be

thus, As $3 : 21 :: 6 : 42$,
or thus, As $6 : 42 :: 3 : 21$.

But the Rule of Three Inverse, is when more requires less, or less requires more. As in this: if 3 men dig a certain quantity of trench in 14 hours, in how many hours will 6 men dig the like quantity? Here it is evident that 6 men, being more than 3, will perform an equal quantity of work in less time, or fewer hours. Or thus: if 6 men perform a certain quantity of work in 7 hours, in how many hours will 3 men perform the same? Here less requires more, for 3 men will take more hours than 6 to perform the same work. In both these cases then the Rule, or the Proportion, is Inverse; and the stating must be

thus, As $6 : 14 :: 3 : 7$,
or thus, As $3 : 7 :: 6 : 14$.

And in all these statings, the fourth term is found, by multiplying the 2d and 3d terms together, and dividing the product by the 1st term.

Of the three given numbers; two of them contain the supposition, and the third a demand. And for stating and working questions of these kinds, observe the following general Rule:

STATE

State the question, by setting down in a straight line the three given numbers, in the following manner, viz. so that the 2d term be that number of supposition which is of the same kind that the answer or 4th term is to be; making the other number of supposition the 1st term, and the demanding number the 3d term, when the question is in direct proportion; but contrariwise, the other number of supposition the 3d term, and the demanding number the 1st term, when the question has inverse proportion.

Then, in both cases, multiply the 2d and 3d terms together, and divide the product by the 1st, which will give the answer, or 4th term sought, viz. of the same denomination as the second term.

Note, If the first and third terms consist of different denominations, reduce them both to the same: and if the second term be a compound number, it is mostly convenient to reduce it to the lowest denomination mentioned.—If, after division, there be any remainder, reduce it to the next lower denomination, and divide by the same divisor as before, and the quotient will be of this last denomination. Proceed in the same manner with all the remainders, till they be reduced to the lowest denomination which the second admits of, and the several quotients taken together will be the answer required.

Note also, The reason for the foregoing Rules will appear, when we come to treat of the nature of Proportions.—Sometimes two or more statings are necessary, which may always be known from the nature of the question.

EXAMPLES.

1. If 8 yards of Cloth cost 1*l* 4*s*, what will 96 yards cost?

```
     yds  l  s    yds   l  s
  As  8 : 1  4 :: 96 : 14  8  the Answer.
            20
           ---
            24
            96
           ---
           144
          216
          ----
      8 ) 2304
          ----
    2,0 ) 28,8s
          ----
         £14  8  Answer.
          ----
```

Ex. 2.

Ex. 2. An engineer having raised 100 yards of a certain work in 24 days with 5 men; how many men must he employ to finish a like quantity of work in 15 days?

ds men ds men
As 15 : 5 :: 24 : 8 Ans.
5

15) 120 (8 Answer.
120

3. What will 72 yards of cloth cost, at the rate of 9 yards for 5*l* 12*s*? Ans. 44*l* 16*s*.

4. A person's annual income being 146*l*; how much is that per day? Ans. 8*s*.

5. If 3 paces or common steps of a certain person be equal to 2 yards, how many yards will 160 of his paces make? Ans. 106 yds 2 ft.

6. What length must be cut off a board, that is 9 inches broad, to make a square foot, or as much as 12 inches in length and 12 in breadth contains? Ans. 16 inches.

7. If 750 men require 22500 rations of bread for a month; how many rations will a garrison of 1200 men require? Ans. 36000.

8. If 7 cwt 1 qr of sugar cost 26*l* 10*s* 4*d*; what will be the price of 43 cwt 2 qrs? Ans. 159*l* 2*s*.

9. The clothing of a regiment of foot of 750 men amounting to 2831*l* 5*s*; what will the clothing of a body of 3500 men amount to? Ans. 13212*l* 10*s*.

10. How many yards of matting, that is 3 ft broad, will cover a floor that is 27 feet long and 20 feet broad? Ans. 60 yards.

11. What is the value of 6 bushels of coals, at the rate of 1*l* 14*s* 6*d* the chaldron? Ans. 5*s* 9*d*.

12. If 6352 stones of 3 feet long complete a certain quantity of walling; how many stones of 2 feet long will raise a like quantity? Ans. 9528.

13. What must be given for a piece of silver weighing 73 lb 5 oz 15 dwts, at the rate of 5*s* 9*d* per ounce? Ans. 253*l* 10*s* $0\frac{3}{4}d$.

14. A garrison of 536 men having provision for 12 months; how long will those provisions last, if the garrison be increased to 1124 men? Ans. 174 days and $\frac{64}{1124}$.

15. What will be the tax upon 763*l* 15*s*, at the rate of 3*s* 6*d* per pound sterling? Ans. 133*l* 13*s* $1\frac{1}{2}d$.

16. A

16. A certain work being raised in 12 days, by working 4 hours each day; how long would it have been in raising by working 6 hours per day? Ans. 8 days.

17. What quantity of corn can I buy for 90 guineas, at the rate of 6s the bushel? Ans. 39 qrs 3 bu.

18. A person, failing in trade, owes in all 977*l*; at which time he has, in money, goods, and recoverable debts, 420*l* 6*s* $3\frac{1}{4}d$; now supposing these things delivered to his creditors, how much will they get per pound? Ans. 8*s* $7\frac{1}{4}d$.

19. A plain of a certain extent having supplied a body of 3000 horse with forage for 18 days; then how many days would the same plain have supplied a body of 2000 horse?

Ans. 27 days.

20. Suppose a gentleman's income is 600 guineas a year, and that he spends 25*s* 6*d* per day, one day with another; how much will he have saved at the year's end?

Ans. 164*l* 12*s* 6*d*.

21. What cost 30 pieces of lead, each weighing 1 cwt 12lb, at the rate of 16*s* 4*d* the cwt? Ans. 27*l* 2*s* 6*d*.

22. The governor of a besieged place having provision for 54 days, at the rate of $1\frac{1}{2}$ lb of bread; but being desirous to prolong the siege to 80 days, in expectation of succour, in that case what must the ration of bread be? Ans. $1\frac{1}{80}$lb.

23. At half a guinea per week, how long can I be boarded for 20 pounds? Ans. $38\frac{12}{126}$ wks.

24. How much will 75 chaldrons 7 bushels of coals come to, at the rate of 1*l* 13*s* 6*d* per chaldron?

Ans. 125*l* 19*s* $0\frac{1}{6}d$.

25. If the penny loaf weigh 8 ounces when the bushel of wheat costs 7*s* 3*d*, what ought the penny loaf to weigh when the wheat is at 8*s* 4*d*? Ans. 6 oz $15\frac{36}{100}$ dr.

26. How much a year will 173 acres 2 roods 14 poles of land give, at the rate of 1*l* 7*s* 8*d* per acre?

Ans. 240*l* 2*s* $7\frac{1}{20}d$.

27. To how much amounts 73 pieces of lead, each weighing 1 cwt 3 qrs 7 lb, at 10*l* 4*s* per fother of $19\frac{1}{2}$ cwt?

Ans. 69*l* 4*s* 2*d* $1\frac{18}{78}$ q.

28. How many yards of stuff, of 3 qrs wide, will line a cloak that is $1\frac{3}{4}$ yards in length and $3\frac{1}{2}$ yards wide?

Ans. 8 yds 0 qrs $2\frac{2}{3}$ nl.

29. If 5 yards of cloth cost 14*s* 2*d*, what must be given for 9 pieces, containing each 21 yards 1 quarter?

Ans. 27*l* 1*s* $10\frac{1}{2}d$.

30. If a gentleman's estate be worth 2107*l* 12*s* a year; what may he spend per day, to save 500*l* in the year?

Ans. 4*l* 8*s* $1\frac{19}{365}d$.

31. Wanting

31. Wanting just an acre of land cut off from a piece which is $13\frac{1}{2}$ poles in breadth, what length must the piece be?
Ans 11 po 4 yds 2 ft $0\frac{18}{27}$ in.

32. At 7*s* $9\frac{1}{2}$*d* per yard, what is the value of a piece of cloth containing 53 ells English 1 qu. Ans. 25*l* 18*s* $1\frac{3}{4}$*d*.

33. If the carriage of 5 cwt 14 lb for 96 miles be 1*l* 12*s* 6*d*; how far may I have 3 cwt 1 qr carried for the same money?
Ans. 151 m 3 fur $3\frac{1}{13}$ pol.

34. Bought a silver tankard, weighing 1 lb 7 oz 14 dwts; what did it cost me at 6*s* 4*d* the ounce? Ans. 6*l* 4*s* $9\frac{1}{3}$*d*.

35. What is the half year's rent of 547 acres of land, at 15*s* 6*d* the acre? Ans. 211*l* 19*s* 3*d*.

36. A wall that is to be built to the height of 36 feet, was raised 9 feet high by 16 men in 6 days; then how many men must be employed to finish the wall in 4 days, at the same rate of working? Ans. 72 men.

37. What will be the charge of keeping 20 horses for a year, at the rate of $14\frac{1}{2}$*d* per day for each horse?
Ans. 441*l* 0*s* 10*d*.

38. If 18 ells of stuff that is $\frac{3}{4}$ yard wide, cost 39*s* 6*d*; what will 50 ells, of the same goodness, cost, being yard wide?
Ans. 7*l* 6*s* $3\frac{5}{9}$*d*.

39. How many yards of paper that is 30 inches wide, will hang a room that is 20 yards in circuit and 9 feet high?
Ans. 72 yards.

40. If a gentleman's estate be worth 384*l* 16*s* a year, and the land-tax be assessed at 2*s* $9\frac{1}{4}$*d* per pound, what is his net annual income? Ans. 331*l* 1*s* $9\frac{1}{3}$*d*.

41. The circumference of the earth is about 25000 miles; at what rate per hour is a person at the middle of its surface carried round, one whole rotation being made in 23 hours 56 minutes? Ans. $1044\frac{816}{1436}$ miles.

42. If a person drink 20 bottles of wine per month, when it costs 8*s* a gallon; how many bottles per month may he drink, without increasing the expense, when wine costs 10*s* the gallon? Ans. 16 bottles.

43. What cost 43 qrs 5 bushels of corn, at 1*l* 8*s* 6*d* the quarter? Ans. 62*l* 3*s* $3\frac{3}{4}$*d*.

44. How many yards of canvas that is ell wide will line 50 yards of say that is 3 quarters wide? Ans. 30 yds.

45. If an ounce of gold cost 4 guineas, what is the value of a grain? Ans. $2\frac{1}{10}$*d*.

46. If 3 cwt of tea cost 40*l* 12*s*; at how much a pound must it be retailed, to gain 10*l* by the whole? Ans. $3\frac{4}{336}$*s*.

COMPOUND

COMPOUND PROPORTION.

Compound Proportion shows how to resolve such questions as require two or more statings by Simple Proportion; and these may be either Direct or Inverse.

In these questions, there is always given an odd number of terms, either five, or seven, or nine, &c. These are distinguished into terms of supposition, and terms of demand, there being always one term more of the former than of the latter, which is of the same kind with the answer sought. The method is thus:

Set down in the middle place that term of supposition which is of the same kind with the answer sought.—Take one of the other terms of supposition, and one of the demanding terms which is of the same kind with it; then place one of them for a first term, and the other for a third, according to the directions given in the Rule of Three.—Do the same with another term of supposition, and its corresponding demanding term; and so on if there be more terms of each kind; setting the numbers under each other which fall all on the left-hand side of the middle term, and the same for the others on the right-hand side.—Then, to work

By several Operations.—Take the two upper terms and the middle term, in the same order as they stand, for the first Rule-of-Three question to be worked, whence will be found a fourth term. Then take this fourth number, so found, for the middle term of a second Rule-of-Three question, and the next two under terms in the general stating, in the same order as they stand, finding a fourth term for them. And so on, as far as there are any numbers in the general stating, making always the fourth number, resulting from each simple stating, to be the second term in the next following one. So shall the last resulting number be the answer to the question.

By one Operation.—Multiply together all the terms standing under each other, on the left-hand side of the middle term; and, in like manner, multiply together all those on the right-hand side of it. Then multiply the middle term by the latter product, and divide the result by the former product; so shall the quotient be the answer sought.

EXAMPLES.

1. How many men can complete a trench of 135 yards long in 8 days, when 16 men can dig 54 yards in 6 days?

General Stating.

```
yds   54 : 16 :: 135 yds
days   8          6 days
     ---        ---
     432        810
     ---         16
                ----
                4860
                 81    men
            ---------
       432 ) 12960 ( 30 Ans. by one operation.
             1296
             -----
                 0
                 --
```

The same by two Operations.

```
         1st.                       2d.
As 54 : 16 :: 135 : 40  |  As 8 : 40 :: 6 : 30
               16       |          6
              ---       |        ---
              810       |   8 ) 240 ( 30 Ans.
             135        |        24
             ---        |        ---
       54 ) 2160 ( 40   |          0
            216         |          -
            ---         |
              0         |
              -         |
```

2. If 100*l* in one year gain 5*l* interest, what will be the interest of 750*l* for 7 years? Ans. 262*l* 10*s*.

3. If a family of 8 persons expend 200*l* in 9 months; how much will serve a family of 18 people 12 months? Ans. 300*l*.

4. If 27*s* be the wages of 4 men for 7 days; what will be the wages of 14 men for 10 days? Ans. 6*l* 15*s*.

5. If a footman travel 130 miles in 3 days, when the days are 12 hours long; in how many days, of 10 hours each, may he travel 360 miles? Ans. $9\frac{63}{65}$ days.

Ex. 6.

Ex. 6. If 120 bushels of corn can serve 14 horses 56 days; how many days will 94 bushels serve 6 horses?

Ans. $102\frac{16}{45}$ days.

7. If 3000 lb of beef serve 340 men 15 days; how many lbs will serve 120 men for 25 days? Ans. 1764 lb $11\frac{5}{17}$ oz.

8. If a barrel of beer be sufficient to last a family of 8 persons 12 days; how many barrels will be drank by 16 persons in the space of a year? Ans. $60\frac{5}{6}$ barrels.

9. If 180 men, in 6 days, of 10 hours each, can dig a trench 200 yards long, 3 wide, and 2 deep; in how many days, of 8 hours long, will 100 men dig a trench of 360 yards long, 4 wide, and 3 deep? Ans. 15 days.

OF VULGAR FRACTIONS.

A Fraction, or broken number, is an expression of a part, or some parts, of something considered as a whole.

It is denoted by two numbers, placed one below the other, with a line between them:

Thus, $\left.\frac{3 \text{ numerator}}{4 \text{ denominator}}\right\}$, which is named 3-fourths.

The Denominator, or number placed below the line, shows how many equal parts the whole quantity is divided into; and it represents the Divisor in Division.—And the Numerator, or number set above the line, shows how many of these parts are expressed by the Fraction: being the remainder after division.—Also, both these numbers are, in general, named the Terms of the Fraction.

Fractions are either Proper, Improper, Simple, Compound, or Mixed.

A Proper Fraction, is when the numerator is less than the denominator; as, $\frac{1}{2}$, or $\frac{2}{3}$, or $\frac{3}{5}$, &c.

An Improper Fraction, is when the numerator is equal to, or exceeds, the denominator; as, $\frac{3}{3}$, or $\frac{5}{4}$, or $\frac{7}{5}$, &c.

A Simple Fraction, is a single expression, denoting any number of parts of the integer; as, $\frac{2}{3}$, or $\frac{3}{2}$.

A Compound Fraction, is the fraction of a fraction, or several fractions connected with the word *of* between them; as, $\frac{1}{2}$ of $\frac{2}{3}$, or $\frac{3}{5}$ of $\frac{5}{6}$ of 3, &c.

A Mixed Number, is composed of a whole number and a fraction together; as, $3\frac{1}{4}$, or $12\frac{4}{5}$, &c.

A whole or integer number may be expressed like a fraction, by writing 1 below it, as a denominator; so 3 is $\frac{3}{1}$, or 4 is $\frac{4}{1}$, &c.

A fraction denotes division; and its value is equal to the quotient obtained by dividing the numerator by the denominator: so $\frac{12}{4}$ is equal to 3, and $\frac{20}{5}$ is equal to 4.

Hence then, if the numerator be less than the denominator, the value of the fraction is less than 1. But if the numerator be the same as the denominator, the fraction is just equal to 1. And if the numerator be greater than the denominator, the fraction is greater than 1.

REDUCTION OF VULGAR FRACTIONS.

Reduction of Vulgar Fractions, is the bringing them out of one form or denomination into another; commonly to prepare them for the operations of Addition, Subtraction, &c, of which there are several cases.

PROBLEM.

To find the Greatest Common Measure of Two or more Numbers.

The Common Measure of two or more numbers, is that number which will divide them both without remainder; so, 3 is a common measure of 18 and 24; the quotient of the former being 6, and of the latter 8. And the greatest number that will do this, is the greatest common measure: so 6 is the greatest common measure of 18 and 24; the quotient of the former being 3, and of the latter 4, which will not both divide further.

RULE.

If there be two numbers only; divide the greater by the less; then divide the divisor by the remainder; and so on, dividing always the last divisor by the last remainder, till nothing remains; so shall the last divisor of all be the greatest common measure sought.

When there are more than two numbers, find the greatest common measure of two of them, as before; then do the same for that common measure and another of the numbers; and

and so on, through all the numbers; so will the greatest common measure last found be the answer.

If it happen that the common measure thus found is 1; then the numbers are said to be incommensurable, or not having any common measure.

EXAMPLES.

1. To find the greatest common measure of 1908, 936, and 630.

```
936 ) 1908 ( 2            So that 36 is the greatest common
      1872                    measure of 1908 and 936.
      ----
        36 ) 936 ( 26  Hence 36 ) 630 ( 17
              72                   36
             ---                  ---
             216                  270
             216                  252
             ---                  ---
                               18 ) 36 ( 2
                                    36
                                    --
```

Hence then 18 is the answer required.

2. What is the greatest common measure of 246 and 372?
Ans. 6.

3. What is the greatest common measure of 324, 612, and 1032?
Ans. 12.

CASE I.

To Abbreviate or Reduce Fractions to their Lowest Terms.

* DIVIDE the terms of the given fraction by any number that will divide them without a remainder; then divide these quotients

* That dividing both the terms of the fraction by the same number, whatever it be, will give another fraction equal to the former, is evident. And when these divisions are performed as often as can be done, or when the common divisor is the greatest possible, the terms of the resulting fraction must be the least possible.

Note. 1. Any number ending with an even number, or a cipher, is divisible, or can be divided, by 2.

2. Any number ending with 5, or 0, is divisible by 5.

3. If

quotients again in the same manner; and so on, till it appears that there is no number greater than 1 which will divide them; then the fraction will be in its lowest terms.

Or, divide both the terms of the fraction by their greatest common measure at once, and the quotients will be the terms of the fraction required, of the same value as at first.

EXAMPLES.

1. Reduce $\frac{216}{288}$ to its least terms.

$\frac{216}{288} = \frac{72}{96} = \frac{36}{48} = \frac{12}{16} = \frac{6}{8} = \frac{3}{4}$, the answer.

Or thus:

```
216) 288 (1
     216
     ---
      72) 216 (3
          216
          ---
```

Therefore 72 is the greatest common measure; and $72)\frac{216}{288} = \frac{3}{4}$ the Answer, the same as before.

2. Reduce

---

3. If the right-hand place of any number be 0, the whole is divisible by 10; if there be two ciphers, it is divisible by 100; if three ciphers, by 1000: and so on; which is only cutting off those ciphers.

4. If the two right-hand figures of any number be divisible by 4, the whole is divisible by 4. And if the three right-hand figures be divisible by 8, the whole is divisible by 8. And so on.

5. If the sum of the digits in any number be divisible by 3, or by 9, the whole is divisible by 3, or by 9.

6. If the right-hand digit be even, and the sum of all the digits be divisible by 6, then the whole is divisible by 6.

7. A number is divisible by 11, when the sum of the 1st, 3d, 5th, &c, or all the odd places, is equal to the sum of the 2d, 4th, 6th, &c, or of all the even places of digits.

8. If a number cannot be divided by some quantity less than the square root of the same, that number is a prime, or cannot be divided by any number whatever.

9. All prime numbers, except 2 and 5, have either 1, 3, 7, or 9, in the place of units; and all other numbers are composite, or can be divided.

10. When

2. Reduce $\frac{195}{780}$ to its lowest terms. Ans. $\frac{1}{4}$.
3. Reduce $\frac{136}{204}$ to its lowest terms. Ans. $\frac{2}{3}$.
4. Reduce $\frac{525}{630}$ to its lowest terms. Ans. $\frac{5}{6}$.

CASE II.

To Reduce a Mixed Number to its Equivalent Improper Fraction.

* MULTIPLY the integer or whole number by the denominator of the fraction, and to the product add the numerator; then set that sum above the denominator for the fraction required.

EXAMPLES.

1. Reduce $23\frac{2}{5}$ to a fraction.

$$\begin{array}{r} 23 \\ 5 \\ \hline 115 \\ 2 \\ \hline 117 \\ \hline 5 \\ \hline \end{array}$$

Or,

$$\frac{(23 \times 5)+2}{5} = \frac{117}{5}, \text{ the Answer.}$$

2. Reduce $12\frac{7}{9}$ to a fraction. Ans. $\frac{115}{9}$.
3. Reduce $14\frac{7}{10}$ to a fraction. Ans. $\frac{147}{10}$.
4. Reduce $183\frac{5}{21}$ to a fraction. Ans. $\frac{3848}{21}$.

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10. When numbers, with the sign of addition or subtraction between them, are to be divided by any number, then each of those numbers must be divided by it. Thus $\frac{10+8-4}{2} = 5 + 4 - 2 = 7$,

11. But if the numbers have the sign of multiplication between them, only one of them must be divided. Thus,

$$\frac{10 \times 8 \times 3}{6 \times 2} = \frac{10 \times 4 \times 3}{6 \times 1} = \frac{10 \times 4 \times 1}{2 \times 1} = \frac{10 \times 2 \times 1}{1 \times 1} = \frac{20}{1} = 20.$$

* This is no more than first multiplying a quantity by some number, and then dividing the result back again by the same: which it is evident does not alter the value; for any fraction represents a division of the numerator by the denominator.

CASE III.

To Reduce an Improper Fraction to its Equivalent Whole or Mixed Number.

* DIVIDE the numerator by the denominator, and the quotient will be the whole or mixed number sought.

EXAMPLES.

1. Reduce $\frac{12}{3}$ to its equivalent number.
   Here $\frac{12}{3}$ or $12 \div 3 = 4$, the Answer.
2. Reduce $\frac{15}{7}$ to its equivalent number.
   Here $\frac{15}{7}$ or $15 \div 7 = 2\frac{1}{7}$, the Answer.
3. Reduce $\frac{749}{17}$ to its equivalent number.

Thus, 17) 749 ($44\frac{1}{17}$
68
——
69 So that $\frac{749}{17} = 44\frac{1}{17}$, the Answer.
68
——
1
——

4. Reduce $\frac{56}{7}$ to its equivalent number. Ans. 8.
5. Reduce $\frac{1362}{25}$ to its equivalent number. Ans. $54\frac{12}{25}$.
6. Reduce $\frac{2918}{17}$ to its equivalent number. Ans. $171\frac{11}{17}$.

CASE IV.

To Reduce a Whole Number to an Equivalent Fraction, having a Given Denominator.

† MULTIPLY the whole number by the given denominator; then set the product over the said denominator, and it will form the fraction required.

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* This Rule is evidently the reverse of the former; and the reason of it is manifest from the nature of Common Division.

† Multiplication and Division being here equally used, the result must be the same as the quantity first proposed.

EXAMPLES.

EXAMPLES.

1. Reduce 9 to a fraction whose denominator shall be 7.

Here $9 \times 7 = 63$: then $\frac{63}{7}$ is the Answer;

For $\frac{63}{7} = 63 \div 7 = 9$, the Proof.

2. Reduce 12 to a fraction whose denominator shall be 13.

Ans. $\frac{156}{13}$.

3. Reduce 27 to a fraction whose denominator shall be 11.

Ans. $\frac{297}{11}$.

CASE V.

To Reduce a Compound Fraction to an Equivalent Simple One.

* MULTIPLY all the numerators together for a numerator, and all the denominators together for a denominator, and they will form the simple fraction sought.

When part of the compound fraction is a whole or mixed number, it must first be reduced to a fraction by one of the former cases.

And, when it can be done, any two terms of the fraction may be divided by the same number, and the quotients used instead of them. Or, when there are terms that are common, they may be omitted, or cancelled.

EXAMPLES.

1. Reduce $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{3}{4}$ to a simple fraction.

Here $\frac{1 \times 2 \times 3}{2 \times 3 \times 4} = \frac{6}{24} = \frac{1}{4}$, the Answer.

Or, $\frac{1 \times \cancel{2} \times \cancel{3}}{\cancel{2} \times \cancel{3} \times 4} = \frac{1}{4}$, by cancelling the 2's and 3's.

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* The truth of this Rule may be shown as follows: Let the compound fraction be $\frac{2}{3}$ of $\frac{5}{7}$. Now $\frac{1}{3}$ of $\frac{5}{7}$ is $\frac{5}{7} \div 3$, which is $\frac{5}{21}$; consequently $\frac{2}{3}$ of $\frac{5}{7}$ will be $\frac{5}{21} \times 2$ or $\frac{10}{21}$; that is, the numerators are multiplied together, and also the denominators, as in the Rule. When the compound fraction consists of more than two single ones; having first reduced two of them as above, then the resulting faction and a third will be the same as a compound fraction of two parts; and so on to the last of all.

2. Reduce

2. Reduce $\frac{2}{3}$ of $\frac{3}{5}$ of $\frac{10}{11}$ to a simple fraction.

Here $\dfrac{2\times3\times10}{3\times5\times11}=\dfrac{60}{165}=\dfrac{12}{33}=\dfrac{4}{11}$, the Answer.

Or, $\dfrac{2\times\cancel{3}\times\cancel{10}}{\cancel{3}\times\cancel{5}\times11}=\dfrac{4}{11}$, the same as before, by cancelling the 3's, and dividing by 5's.

3. Reduce $\frac{3}{7}$ of $\frac{4}{5}$ to a simple fraction. Ans. $\frac{12}{35}$.
4. Reduce $\frac{2}{3}$ of $\frac{3}{5}$ of $\frac{5}{9}$ to a simple fraction. Ans. $\frac{2}{9}$.
5. Reduce $\frac{2}{5}$ of $\frac{5}{8}$ of $3\frac{1}{2}$ to a simple fraction. Ans. $\frac{7}{8}$.
6. Reduce $\frac{2}{7}$ of $\frac{5}{8}$ of $\frac{7}{2}$ of 4 to a simple fraction. Ans. $\frac{5}{2}$.
7. Reduce 2 and $\frac{2}{5}$ of $\frac{5}{6}$ to a fraction. Ans. $\frac{7}{3}$.

CASE VI.

To Reduce Fractions of Different Denominators, to Equivalent Fractions having a Common Denominator.

* MULTIPLY each numerator by all the denominators except its own, for the new numerators: and multiply all the denominators together for a common denominator.

Note, It is evident, that in this and several other operations, when any of the proposed quantities are integers, or mixed numbers, or compound fractions, they must first be reduced, by their proper Rules, to the form of simple fractions.

EXAMPLES.

1. Reduce $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$, to a common denominator.

$1 \times 3 \times 4 = 12$ the new numerator for $\frac{1}{2}$.
$2 \times 2 \times 4 = 16$ ditto $\frac{2}{3}$.
$3 \times 2 \times 3 = 18$ ditto $\frac{3}{4}$.
$2 \times 3 \times 4 = 24$ the common denominator.

Therefore the equivalent fractions are $\frac{12}{24}$, $\frac{16}{24}$, and $\frac{18}{24}$.

Or the whole operation of multiplying may be best performed mentally, only setting down the results and given fractions thus: $\frac{1}{2}, \frac{2}{3}, \frac{3}{4} = \frac{12}{24}, \frac{16}{24}, \frac{18}{24} = \frac{6}{12}, \frac{8}{12}, \frac{9}{12}$, by abbreviation.

2. Reduce $\frac{2}{7}$ and $\frac{5}{9}$ to fractions of a common denominator. Ans. $\frac{18}{63}$, $\frac{35}{63}$.

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* This is evidently no more than multiplying each numerator and its denominator by the same quantity, and consequently the value of the fraction is not altered.

3. Reduce

3. Reduce $\frac{2}{3}$, $\frac{3}{5}$, and $\frac{3}{4}$ to a common denominator.

Ans. $\frac{40}{60}$, $\frac{36}{60}$, $\frac{45}{60}$.

4. Reduce $\frac{5}{6}$, $2\frac{3}{5}$, and 4 to a common denominator.

Ans. $\frac{25}{30}$, $\frac{78}{30}$, $\frac{120}{30}$.

Note 1. When the denominators of two given fractions have a common measure, let them be divided by it; then multiply the terms of each given fraction by the quotient arising from the other's denominator.

Ex. $\underset{5}{\frac{3}{25}}$ and $\underset{7}{\frac{4}{35}} = \frac{21}{175}$ and $\frac{20}{175}$, by multiplying the former by 7 and the latter by 5.

2. When the less denominator of two fractions exactly divides the greater, multiply the terms of that which has the less denominator by the quotient.

Ex. $\frac{3}{7}$ and $\underset{2}{\frac{5}{14}} = \frac{6}{14}$ and $\frac{5}{14}$, by mult. the former by 2.

3. When more than two fractions are proposed, it is sometimes convenient, first to reduce two of them to a common denominator; then these and a third; and so on till they be all reduced to their least common denominator.

Ex. $\frac{2}{3}$ and $\frac{3}{4}$ and $\frac{7}{8} = \frac{2}{3}$ and $\frac{6}{8}$ and $\frac{7}{8} = \frac{16}{24}$ and $\frac{18}{24}$ and $\frac{21}{24}$.

CASE VII.

To find the value of a Fraction in Parts of the Integer.

MULTIPLY the integer by the numerator, and divide the product by the denominator, by Compound Multiplication and Division, if the integer be a compound quantity.

Or, if it be a single integer, multiply the numerator by the parts in the next inferior denomination, and divide the product by the denominator. Then, if any thing remains, multiply it by the parts in the next inferior denomination, and divide by the denominator as before; and so on as far as necessary; so shall the quotients, placed in order, be the value of the fraction required *.

* The numerator of a fraction being considered as a remainder, in Division, and the denominator as the divisor, this rule is of the same nature as Compound Division, or the valuation of remainders in the Rule of Three, before explained.

EXAMPLES.

EXAMPLES.

1. What is the $\frac{4}{5}$ of 2*l* 6*s*?

By the former part of the Rule,

$$\begin{array}{r} 2l\ 6s \\ 4 \\ \hline 5)\ 9\ \ 4 \\ \end{array}$$

Ans. 1*l* 16*s* 9*d* $2\frac{2}{5}q$.

2. What is the value of $\frac{2}{3}$ of 1*l*?

By the 2d part of the Rule,

$$\begin{array}{l} \ \ \ \ 2 \\ \ \ \ 20 \\ \hline 3)\ 40\ (13s\ 4d\ \text{Ans.} \\ \hline \ \ \ \ \ 1 \\ \ \ \ \ 12 \\ \hline 3)\ 12\ (4d \\ \hline \end{array}$$

3. Find the value of $\frac{3}{8}$ of a pound sterling. Ans. 7*s* 6*d*.
4. What is the value of $\frac{2}{9}$ of a guinea? Ans. 4*s* 8*d*.
5. What is the value of $\frac{3}{4}$ of a half crown? Ans. 1*s* $10\frac{1}{2}d$.
6. What is the value of $\frac{2}{5}$ of 4*s* 10*d*? Ans. 1*s* $11\frac{1}{5}d$.
7. What is the value of $\frac{4}{5}$ lb troy? Ans. 9 oz 12 dwts.
8. What is the value of $\frac{5}{16}$ of a cwt? Ans. 1 qr 7 lb.
9. What is the value of $\frac{7}{8}$ of an acre? Ans. 3 ro. 20 po.
10. What is the value of $\frac{3}{10}$ of a day? Ans. 7 hrs 12 min.

CASE VIII.

To Reduce a Fraction from one Denomination to another.

* CONSIDER how many of the less denomination make one of the greater; then multiply the numerator by that number, if the reduction be to a less name, but multiply the denominator, if to a greater.

EXAMPLES.

1. Reduce $\frac{2}{9}$ of a pound to the fraction of a penny.

$\frac{2}{9} \times \frac{20}{1} \times \frac{12}{1} = \frac{480}{9} = \frac{160}{3}$, the Answer.

* This is the same as the Rule of Reduction in whole numbers from one denomination to another.

2. Reduce

2. Reduce $\frac{5}{7}$ of a penny to the fraction of a pound.
$\frac{5}{7} \times \frac{1}{12} \times \frac{1}{20} = \frac{1}{336}$, the Answer.
3. Reduce $\frac{2}{13}l$ to the fraction of a penny. Ans. $\frac{32}{1}d$.
4. Reduce $\frac{2}{5}$q to the fraction of a pound. Ans. $\frac{1}{2400}$.
5. Reduce $\frac{2}{7}$ cwt to the fraction of a lb. Ans. $\frac{32}{1}$.
6. Reduce $\frac{3}{5}$ dwt to the fraction of a lb troy. Ans. $\frac{1}{400}$.
7. Reduce $\frac{3}{8}$ crown to the fraction of a guinea. Ans. $\frac{5}{56}$.
8. Reduce $\frac{5}{6}$ half-crown to the fract. of a shilling. Ans. $\frac{25}{12}$.
9. Reduce 2*s* 6*d* to the fraction of a £. Ans. $\frac{1}{8}$.
10. Reduce 17*s* 7*d* $3\frac{3}{5}q$ to the fraction of a £.

ADDITION OF VULGAR FRACTIONS.

If the fractions have a common denominator; add all the numerators together, then place the sum over the common denominator, and that will be the sum of the fractions required.

* If the proposed fractions have not a common denominator, they must be reduced to one. Also compound fractions must be reduced to simple ones, and fractions of different denominations to those of the same denomination. Then add the numerators as before. As to mixed numbers, they may either be reduced to improper fractions, and so added with the others; or else the fractional parts only added, and the integers united afterwards.

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* Before fractions are reduced to a common denominator, they are quite dissimilar, as much as shillings and pence are, and therefore cannot be incorporated with one another, any more than these can. But when they are reduced to a common denominator, and made parts of the same thing, their sum, or difference, may then be as properly expressed by the sum or difference of the numerators, as the sum or difference of any two quantities whatever, by the sum or difference of their individuals. Whence the reason of the Rule is manifest, both for Addition and Subtraction.

When several fractions are to be collected, it is commonly best first to add two of them together that most easily reduce to a common denominator; then add th[illegible]d a third, and so on.

EXAMPLES.

EXAMPLES.

1. To add $\frac{3}{5}$ and $\frac{4}{5}$ together.
   Here $\frac{3}{5}+\frac{4}{5}=\frac{7}{5}=1\frac{2}{5}$, the Answer.
2. To add $\frac{3}{5}$ and $\frac{5}{6}$ together.
   $\frac{3}{5}+\frac{5}{6}=\frac{18}{30}+\frac{25}{30}=\frac{43}{30}=1\frac{13}{30}$, the Answer.
3. To add $\frac{5}{8}$ and $7\frac{1}{2}$ and $\frac{1}{3}$ of $\frac{3}{4}$ together.
   $\frac{5}{8}+7\frac{1}{2}+\frac{1}{3}$ of $\frac{3}{4}=\frac{5}{8}+\frac{15}{2}+\frac{1}{4}=\frac{5}{8}+\frac{60}{8}+\frac{2}{8}=\frac{67}{8}=8\frac{3}{8}$.
4. To add $\frac{3}{7}$ and $\frac{6}{7}$ together. Ans. $1\frac{2}{7}$.
5. To add $\frac{3}{4}$ and $\frac{5}{9}$ together. Ans. $1\frac{11}{36}$.
6. Add $\frac{2}{7}$ and $\frac{5}{14}$ together. Ans. $\frac{9}{14}$.
7. What is the sum of $\frac{2}{3}$ and $\frac{3}{5}$ and $\frac{5}{7}$? Ans. $1\frac{103}{105}$.
8. What is the sum of $\frac{5}{9}$ and $\frac{3}{5}$ and $2\frac{1}{6}$? Ans. $3\frac{29}{90}$.
9. What is the sum of $\frac{3}{5}$ and $\frac{4}{5}$ of $\frac{1}{3}$, and $9\frac{3}{20}$? Ans. $10\frac{1}{60}$.
10. What is the sum of $\frac{2}{3}$ of a pound and $\frac{5}{9}$ of a shilling?
    Ans. $\frac{125}{9}s$ or $13s\ 10d\ 2\frac{2}{3}q$.
11. What is the sum of $\frac{3}{5}$ of a shilling and $\frac{4}{15}$ of a penny?
    Ans. $\frac{112}{15}d$ or $7d\ 1\frac{13}{15}q$.
12. What is the sum of $\frac{1}{7}$ of a pound, and $\frac{2}{9}$ of a shilling, and $\frac{5}{12}$ of a penny? Ans. $\frac{3139}{1008}s$ or $3s\ 1d\ 1\frac{10}{21}q$.

SUBTRACTION OF VULGAR FRACTIONS.

PREPARE the fractions the same as for Addition, when necessary; then subtract the one numerator from the other, and set the remainder over the common denominator, for the difference of the fractions sought.

EXAMPLES.

1. To find the difference between $\frac{5}{6}$ and $\frac{1}{6}$.
   Here $\frac{5}{6}-\frac{1}{6}=\frac{4}{6}=\frac{2}{3}$, the Answer.
2. To find the difference between $\frac{3}{4}$ and $\frac{5}{9}$.
   $\frac{3}{4}-\frac{5}{9}=\frac{27}{36}-\frac{20}{36}=\frac{7}{36}$, the Answer.

3. What

3. What is the difference between $\frac{5}{12}$ and $\frac{7}{12}$? Ans. $\frac{1}{6}$.

4. What is the difference between $\frac{3}{13}$ and $\frac{4}{39}$? Ans. $\frac{5}{39}$.

5. What is the difference between $\frac{5}{12}$ and $\frac{7}{13}$? Ans. $\frac{19}{156}$.

6. What is the diff. between $5\frac{3}{8}$ and $\frac{2}{7}$ of $4\frac{1}{6}$? Ans. $4\frac{31}{168}$.

7. What is the difference between $\frac{5}{9}$ of a pound, and $\frac{2}{3}$ of $\frac{3}{4}$ of a shilling? Ans. $\frac{191}{18}s$ or $10s\ 7d\ 1\frac{1}{3}q$.

8. What is the difference between $\frac{2}{7}$ of $5\frac{1}{6}$ of a pound, and $\frac{3}{5}$ of a shilling? Ans. $\frac{3037}{2100}l$ or $1l\ 8s\ 11\frac{3}{35}d$.

MULTIPLICATION OF VULGAR FRACTIONS.

* REDUCE mixed numbers, if there be any, to equivalent fractions; then multiply all the numerators together for a numerator, and all the denominators together for a denominator, which will give the product required.

EXAMPLES.

1. Required the product of $\frac{3}{4}$ and $\frac{2}{9}$.

Here $\frac{3}{4} \times \frac{2}{9} = \frac{6}{36} = \frac{1}{6}$, the Answer.

Or $\frac{3}{4} \times \frac{2}{9} = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$.

2. Required the continued product of $\frac{2}{3}$, $3\frac{1}{4}$, 5, and $\frac{3}{4}$ of $\frac{3}{5}$.

$$\text{Here } \frac{\cancel{2}}{3} \times \frac{13}{4} \times \frac{\cancel{5}}{1} \times \frac{\cancel{3}}{4} \times \frac{3}{\cancel{5}} = \frac{13 \times 3}{4 \times 2} = \frac{39}{8} = 4\frac{7}{8}, \text{ Ans.}$$

3. Required the product of $\frac{2}{7}$ and $\frac{5}{8}$. Ans. $\frac{5}{28}$.

4. Required the product of $\frac{4}{15}$ and $\frac{5}{24}$. Ans. $\frac{1}{18}$.

5. Required the product of $\frac{3}{7}$, $\frac{4}{9}$, and $\frac{14}{15}$. Ans. $\frac{8}{45}$.

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* Multiplication of any thing by a fraction, implies the taking some part or parts of the thing; it may therefore be truly expressed by a compound fraction; which is resolved by multiplying together the numerators and the denominators.

Note, A Fraction is best multiplied by an integer, by dividing the denominator by it; but if it will not exactly divide, then multiply the numerator by it.

6. Required

6. Required the product of $\frac{1}{2}$, $\frac{2}{3}$, and 3. Ans. 1.
7. Required the product of $\frac{7}{9}$, $\frac{3}{5}$, and $4\frac{5}{14}$. Ans. $2\frac{1}{30}$.
8. Required the product of $\frac{5}{6}$, and $\frac{2}{3}$ of $\frac{6}{7}$. Ans. $\frac{10}{21}$.
9. Required the product of 6, and $\frac{2}{3}$ of 5. Ans. 20.
10. Required the product of $\frac{2}{9}$ of $\frac{4}{5}$, and $\frac{5}{8}$ of $3\frac{2}{7}$. Ans. $\frac{23}{84}$.
11. Required the product of $3\frac{2}{7}$ and $4\frac{14}{33}$. Ans. $14\frac{124}{231}$.
12. Required the product of 5, $\frac{2}{3}$, $\frac{2}{7}$ of $\frac{3}{5}$, and $4\frac{1}{6}$. Ans. $2\frac{8}{21}$.

DIVISION OF VULGAR FRACTIONS.

* PREPARE the fractions as before in Multiplication; then divide the numerator by the numerator, and the denominator by the denominator, if they will exactly divide: but if not, then invert the terms of the divisor, and multiply the dividend by it, as in Multiplication.

EXAMPLES.

1. Divide $\frac{25}{9}$ by $\frac{5}{3}$.
   Here $\frac{25}{9} \div \frac{5}{3} = \frac{5}{3} = 1\frac{2}{3}$, by the first method.
2. Divide $\frac{5}{9}$ by $\frac{2}{15}$.
   Here $\frac{5}{9} \div \frac{2}{15} = \frac{5}{9} \times \frac{15}{2} = \frac{5}{3} \times \frac{5}{2} = \frac{25}{6} = 4\frac{1}{6}$.
3. It is required to divide $\frac{16}{25}$ by $\frac{4}{5}$. Ans. $\frac{4}{5}$.
4. It is required to divide $\frac{7}{16}$ by $\frac{3}{4}$. Ans. $\frac{7}{12}$.
5. It is required to divide $\frac{14}{9}$ by $\frac{7}{6}$. Ans. $1\frac{1}{3}$.
6. It is required to divide $\frac{5}{6}$ by $\frac{15}{7}$. Ans. $\frac{7}{18}$.
7. It is required to divide $\frac{12}{35}$ by $\frac{3}{5}$. Ans. $\frac{4}{7}$.
8. It is required to divide $\frac{2}{7}$ by $\frac{3}{5}$. Ans. $\frac{10}{21}$.

* Divison being the reverse of Multiplication, the reason of the Rule is evident.

Note, A fraction is best divided by an integer, by dividing the numerator by it; but if it will not exactly divide, then multiply the denominator by it.

9. It is required to divide $\frac{9}{16}$ by 3. Ans. $\frac{3}{16}$.

10. It is required to divide $\frac{3}{5}$ by 2. Ans. $\frac{3}{10}$.

11. It is required to divide $7\frac{1}{3}$ by $9\frac{5}{9}$. Ans. $\frac{33}{43}$.

12. It is required to divide $\frac{2}{3}$ of $\frac{1}{3}$ by $\frac{5}{7}$ of $7\frac{3}{5}$. Ans. $\frac{7}{171}$.

RULE OF THREE IN VULGAR FRACTIONS.

MAKE the necessary preparations as before directed; then multiply continually together, the second and third terms, and the first with its parts inverted as in Division, for the answer*.

EXAMPLES.

1. If $\frac{3}{8}$ of a yard of velvet cost $\frac{2}{5}$ of a pound sterling; what will $\frac{5}{16}$ of a yard cost?

$$\frac{3}{8} : \frac{2}{5} :: \frac{5}{16} : \frac{8}{3} \times \frac{\cancel{2}}{\cancel{5}} \times \frac{\cancel{5}}{\cancel{16}} = \tfrac{1}{3}l = 6s\ 8d, \text{ Answer.}$$

2. What will $3\frac{3}{8}$ oz of silver cost, at 6*s* 4*d* an ounce? Ans. 1*l* 1*s* $4\frac{1}{2}d$.

3. If $\frac{3}{16}$ of a ship be worth 273*l* 2*s* 6*d*; what are $\frac{6}{32}$ of her worth? Ans. 227*l* 12*s* 1*d*.

4. What is the purchase of 1230*l* bank-stock, at $108\frac{5}{8}$ per cent.? Ans. 1336*l* 1*s* 9*d*.

5. What is the interest of 273*l* 15*s* for a year, at $3\frac{1}{4}$ per cent.? Ans. 8*l* 17*s* $11\frac{1}{4}d$.

6. If $\frac{1}{8}$ of a ship be worth 73*l* 1*s* 3*d*; what part of her is worth 250*l* 10*s*? Ans. $\frac{3}{7}$.

7. What length must be cut off a board that is $7\frac{3}{4}$ inches broad, to contain a square foot, or as much as another piece, of 12 inches long and 12 broad? Ans. $18\frac{13}{31}$ inches.

8. What quantity of shalloon that is $\frac{3}{4}$ of a yard wide, will line $9\frac{1}{2}$ yards of cloth, that is $2\frac{1}{2}$ yards wide? Ans. $31\frac{2}{3}$ yds.

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* This is only multiplying the 2d and 3d terms together, and dividing the product by the first, as in the Rule of Three in whole numbers.

9. If the penny loaf weigh $6\frac{9}{10}$ oz, when the price of wheat is 5*s* the bushel; what ought it to weigh when the wheat is 8*s* 6*d* the bushel? Ans. $4\frac{1}{17}$ oz.

10. How much in length, of a piece of land that is $11\frac{11}{12}$ poles broad, will make an acre of land, or as much as 40 poles in length and 4 in breadth? Ans. $13\frac{61}{143}$ poles.

11. If a courier perform a certain journey in $35\frac{1}{2}$ days, travelling $13\frac{5}{8}$ hours a day; how long would he be in performing the same, travelling only $11\frac{9}{10}$ hours a day?
Ans. $40\frac{615}{952}$ days.

12. A regiment of soldiers, consisting of 976 men, are to be new cloathed; each coat to contain $2\frac{1}{2}$ yards of cloth that is $1\frac{5}{8}$ yard wide, and lined with shalloon $\frac{7}{8}$ yard wide: how many yards of shalloon will line them?
Ans. 4531 yds 1 qr $2\frac{6}{7}$ nails.

DECIMAL FRACTIONS.

A Decimal Fraction, is that which has for its denominator an unit (1), with as many ciphers annexed as the numerator has places; and it is usually expressed by setting down the numerator only, with a point before it, on the left-hand. Thus, $\frac{4}{10}$ is ·4, and $\frac{24}{100}$ is ·24, and $\frac{74}{1000}$ is ·074, and $\frac{124}{100000}$ is ·00124; where ciphers are prefixed to make up as many places as are ciphers in the denominator, when there is a deficiency of figures.

A mixed number is made up of a whole number with some decimal fraction, the one being separated from the other by a point. Thus, 3·25 is the same as $3\frac{25}{100}$, or $\frac{325}{100}$.

Ciphers on the right-hand of decimals make no alteration in their value; for ·4, or ·40, or ·400 are decimals having all the same value, each being $= \frac{4}{10}$, or $\frac{2}{5}$. But when they are placed on the left-hand, they decrease the value in a ten-fold proportion: Thus, ·4 is $\frac{4}{10}$, or 4 tenths; but ·04 is only $\frac{4}{100}$, or 4 hundredths, and ·004 is only $\frac{4}{1000}$, or 4 thousandths.

The 1st place of decimals, counted from the left-hand towards the right, is called the place of primes, or 10ths; the 2d is the place of seconds, or 100ths; the 3d is the place of thirds, or 1000ths; and so on. For, in decimals, as well as in whole numbers, the values of the places increase towards the left-hand, and decrease towards the right, both in the same

same tenfold proportion; as in the following Scale or Table of Notation.

millions	hundred thousands	ten thousands	thousands	hundreds	tens	units		tenth parts	hundredth parts	thousandth parts	ten thousandth parts	hundred thousandth parts	millionth parts
3	3	3	3	3	3	3	·	3	3	3	3	3	3

ADDITION OF DECIMALS.

SET the numbers under each other according to the value of their places, like as in whole numbers; in which state the decimal separating points will stand all exactly under each other. Then, beginning at the right-hand, add up all the columns of numbers as in integers; and point off as many places, for decimals, as are in the greatest number of decimal places in any of the lines that are added; or place the point directly below all the other points.

EXAMPLES.

1. To add together 29·0146, and 3146·5, and 2109, and ·62417, and 14·16.

```
  29·0146
3146·5
2109·
    ·62417
  14·16
---------
5299·29877 the Sum.
```

Ex. 2. What is the sum of 276, 39·213, 72014·9, 417, and 5032?

3. What is the sum of 7530, 16·201, 3·0142, 957·13, 6·72119 and ·03014.

4. What is the sum of 312·09, 3·5711, 7195·6, 71·498, *9739.215, 179, and ·0027?*

SUBTRACTION OF DECIMALS.

PLACE the numbers under each other according to the value of their places, as in the last Rule. Then, beginning at the right-hand, subtract as in whole numbers, and point off the decimals as in Addition.

EXAMPLES.

1. To find the difference between 91·73 and 2·138.

```
      91·73
       2·138
      ------
Ans.  89·592 the Difference.
```

2. Find the diff. between 1·9185 and 2·73. Ans. 0·8115.
3. To subtract 4·90142 from 214·81. Ans. 209·90858.
4. Find the diff. between 2714 and ·916. Ans. 2713·084.

MULTIPLICATION OF DECIMALS.

* PLACE the factors, and multiply them together the same as if they were whole numbers.—Then point off in the product just as many places of decimals as there are decimals in both the factors. But if there be not so many figures in the product, then supply the defect by prefixing ciphers.

* The Rule will be evident from this example:—Let it be required to multiply ·12 by ·361; these numbers are equivalent to $\frac{12}{100}$ and $\frac{361}{1000}$; the product of which is $\frac{4332}{100000}$ = ·04332, by the nature of Notation, which consists of as many places as there are ciphers, that is, of as many places as there are in both numbers. And in like manner for any other numbers.

EXAMPLES.

EXAMPLES.

1. Multiply ·3210906
by ·2465

1605480
1926576
1284384
642192

Ans. ·0791501640 the Product.

2. Multiply 79·347 by 23·15. Ans. 1836·88305.
3. Multiply ·63478 by ·8204. Ans. .520773512.
4. Multiply ·385746 by ·00464. Ans. ·00178986144.

CONTRACTION I.

To multiply Decimals by 1 *with any number of Ciphers, as by* 10, *or* 100, *or* 1000, *&c.*

THIS is done by only removing the decimal point so many places farther to the right-hand, as there are ciphers in the multiplier; and subjoining ciphers if need be.

EXAMPLES.

1. The product of 51·3 and 1000 is 51300.
2. The product of 2·714 and 100 is
3. The product of ·916 and 1000 is
4. The product of 21·31 and 10000 is

CONTRACTION II.

To Contract the Operation, so as to retain only as many Decimals in the Product as may be thought Necessary, when the Product would naturally contain several more Places.

SET the units' place of the multiplier under that figure of the multiplicand whose place is the same as is to be retained for the last in the product; and dispose of the rest of the figures in the inverted or contrary order to what they are usually placed in.—Then, in multiplying, reject all the figures that are more to the right-hand than each multiplying figure, and set down the products, so that their right-hand figures

may

may fall in a column straight below each other; but observing to increase the first figure of every line with what would arise from the figures omitted, in this manner, namely 1 from 5 to 14, 2 from 15 to 24, 3 from 25 to 34, &c; and the sum of all the lines will be the product as required, commonly to the nearest unit in the last figure.

EXAMPLES.

1. To multiply 27·14986 by 92·41035, so as to retain only four places of decimals in the product.

Contracted Way.	*Common Way.*	
27·14986	27·14986	
53014·29	92·41035	
24434874	13	574930
542997	81	44958
108599	2714	986
2715	108599	44
81	542997	2
14	24434874	
2508·9280	2508·9280	650510

2. Multiply 480·14936 by 2·72416, retaining only four decimals in the product.

3. Multiply 2490·3048 by ·573286, retaining only five decimals in the product.

4. Multiply 325·701428 by ·7218393, retaining only three decimals in the product.

DIVISION OF DECIMALS.

Divide as in whole numbers; and point off in the quotient as many places for decimals, as the decimal places in the dividend exceed those in the divisor*.

* The reason of this Rule is evident; for, since the divisor multiplied by the quotient gives the dividend, therefore the number of decimal places in the dividend, is equal to those in the divisor and quotient, taken together, by the nature of Multiplication; and consequently the quotient itself must contain as many as the dividend exceeds the divisor.

Another

Another way to know the place for the decimal point, is this: The first figure of the quotient must be made to occupy the same place, of integers or decimals, as doth that figure of the dividend which stands over the unit's figure of the first product.

When the places of the quotient are not so many as the Rule requires, the defect is to be supplied by prefixing ciphers.

When there happens to be a remainder after the division; or when the decimal places in the divisor are more than those in the dividend; then ciphers may be annexed to the dividend, and the quotient carried on as far as required.

EXAMPLES.

1.	2.
178) ·48520998 (·00272589	·2639) 27·00000 (102·3114
1292	6100
460	8220
1049	3030
1599	3910
1758	12710
156	2154

3. Divide 123·70536 by 54 25. Ans. 2·2802.
4. Divide 12 by ·7854. Ans. 15·278.
5. Divide 4195·68 by 100. Ans. 41·9568.
6. Divide ·8297592 by ·153. Ans. 5·4232.

CONTRACTION I.

WHEN the divisor is an integer, with any number of ciphers annexed: cut off those ciphers, and remove the decimal point in the dividend as many places farther to the left as there are ciphers cut off, prefixing ciphers if need be; then proceed as before*.

* This is no more than dividing both divisor and dividend by the same number, either 10, or 100, or 1000, &c, according to the number of ciphers cut-off, which, leaving them in the same proportion, does not affect the quotient. And, in the same way, the decimal point may be moved the same number of places in both the divisor and dividend, either to the right or left, whether they have ciphers or not.

EXAMPLES.

EXAMPLES.

1. Divide 45·5 by 2100.

```
21·00) ·455 (·0216, &c.
        35
        140
         14
```

2. Divide 41020 by 32000.
3. Divide 953 by 21600.
4. Divide 61 by 79000.

CONTRACTION II.

HENCE, if the divisor be 1 with ciphers, as 10, 100, or 1000, &c: then the quotient will be found by merely moving the decimal point in the dividend so many places farther to the left, as the divisor has ciphers; prefixing ciphers if need be.

EXAMPLES.

So, 217·3 ÷ 100 = 2·173 And 419 ÷ 10 =
And 5·16 ÷ 100 = And ·21 ÷ 1000 =

CONTRACTION III.

WHEN there are many figures in the divisor; or when only a certain number of decimals are necessary to be retained in the quotient; then take only as many figures of the divisor as will be equal to the number of figures, both integers and decimals, to be in the quotient, and find how many times they may be contained in the first figures of the dividend, as usual.

Let each remainder be a new dividend; and for every such dividend, leave out one figure more on the right-hand side of the divisor; remembering to carry for the increase of the figures cut off, as in the 2d contraction in Multiplication.

Note. When there are not so many figures in the divisor, as are required to be in the quotient, begin the operation with all the figures, and continue it as usual till the number of figures in the divisor be equal to those remaining to be found in the quotient; after which begin the contraction.

EXAMPLES.

1. Divide 2508·92806 by 92·41035, so as to have only four decimals in the quotient, in which case the quotient will contain six figures.

Contracted.

Contracted.	*Common.*
92·4103,5)2508·928,06(27·1498	92·4103,5)2508·928,06(27.1498
660721	66072106
13849	13848610
4608	46075750
912	91116100
80	79467850
6	5539570

2. Divide 4109·2351 by 230·409, so that the quotient may contain only four decimals. Ans. 17·8845.

3. Divide 37·10438 by 5713·96, that the quotient may contain only five decimals. Ans. ·00649.

4. Divide 913·08 by 2137·2, that the quotient may contain only three decimals.

REDUCTION OF DECIMALS.

CASE I.

To reduce a Vulgar Fraction to its equivalent Decimal.

Divide the numerator by the denominator as in Division of Decimals, annexing ciphers to the numerator as far as necessary; so shall the quotient be the decimal required.

EXAMPLES.

1. Reduce $\frac{7}{24}$ to a decimal.

$$24 = 4 \times 6. \quad \text{Then } 4\,)\,7\cdot$$
$$6\,)\,1\cdot750000.$$
$$\cdot291666 \text{ \&c.}$$

2. Reduce $\frac{1}{4}$, and $\frac{1}{2}$, and $\frac{3}{4}$, to decimals. Ans. ·25, and ·5, and ·75.

3. Reduce $\frac{5}{8}$ to a decimal. Ans. ·625.

4. Reduce $\frac{3}{25}$ to a decimal. Ans. ·12.

5. Reduce $\frac{6}{192}$ to a decimal. Ans. ·031350.

6. Reduce $\frac{550}{3842}$ to a decimal. Ans. ·143155 &c.

CASE II.

To find the Value of a Decimal in terms of the Inferior Denominations.

MULTIPLY the decimal by the number of parts in the next lower denomination; and cut off as many places for a remainder to the right-hand, as there are places in the given decimal.

Multiply that remainder by the parts in the next lower denomination again, cutting off for another remainder as before.

Proceed in the same manner through all the parts of the integer; then the several denominations separated on the left-hand, will make up the answer.

Note, This operation is the same as Reduction Descending in whole numbers.

EXAMPLES.

1. Required to find the value of ·775 pounds sterling.

$$\begin{array}{r} \cdot775 \\ 20 \\ \hline s\ 15 \cdot 500 \\ 12 \\ \hline d\ 6 \cdot 000 \\ \hline \end{array} \qquad \text{Ans. } 15s\ 6d.$$

2. What is the value of ·625 shil? Ans. $7\frac{1}{2}d.$
3. What is the value of ·8635*l*? Ans. 17*s* 3·24*d*.
4. What is the value of ·0125 lb troy? Ans. 3 dwts.
5. What is the value of ·4694 lb troy? Ans. 5 oz 12 dwts 15·744 gr.
6. What is the value of ·625 cwt? Ans. 2 qr 14 lb.
7. What is the value of ·009943 miles? Ans. 17 yd 1 ft 5·93848 inc.
8. What is the value of ·6875 yd? Ans. 2 qr 3 nls.
9. What is the value of ·3375 acr? Ans. 1 rd 14 poles.
10. What is the value of ·2083 hhd of wine? Ans. 13·1229 gal.

CASE III.

To reduce Integers or Decimals to Equivalent Decimals of Higher Denominations.

DIVIDE by the number of parts in the next higher denomination; continuing the operation to as many higher denominations as may be necessary, the same as in Reduction Ascending of whole numbers.

EXAMPLES.

1. Reduce 1 dwt to the decimal of a pound troy.

20	1 dwt·
12	0·05 oz
	0·004166 &c. lb. Ans.

2. Reduce 9*d* to the decimal of a pound. Ans. .0375*l.*
3. Reduce 7 drams to the decimal of a pound avoird. Ans. ·02734375lb.
4. Reduce ·26*d* to the decimal of a *l.* Ans. ·0010833 &c. *l.*
5. Reduce 2·15 lb to the decimal of a cwt. Ans. ·019196 + cwt.
6. Reduce 24 yards to the decimal of a mile. Ans. ·013636 &c. mile.
7. Reduce ·056 pole to the decimal of an acre. Ans. ·00035 ac.
8. Reduce 1·2 pint of wine to the decimal of a hhd. Ans. ·00238 + hhd.
9. Reduce 14 minutes to the decimal of a day. Ans. ·009722 &c. da.
10. Reduce ·21 pint to the decimal of a peck. Ans. ·013125 pec.
11. Reduce 28″ 12‴ to the decimal of a minute.

NOTE, *When there are several numbers, to be reduced all to the decimal of the highest:*

Set the given numbers directly under each other, for dividends, proceeding orderly from the lowest denomination to the highest.

Opposite to each dividend, on the left-hand, set such a number for a divisor as will bring it to the next higher name; drawing a perpendicular line between all the divisors and dividends.

Begin at the uppermost, and perform all the divisions: only observing to set the quotient of each division, as decimal parts,

parts, on the right-hand of the dividend next below it; so shall the last quotient be the decimal required.

EXAMPLES.

1. Reduce 17*s* 9$\frac{3}{4}$*d* to the decimal of a pound.

4	3·
12	9·75
20	17·8125
	£ 0 890625 Ans.

2. Reduce 19*l* 17*s* 3$\frac{1}{4}$*d* to *l*. Ans. 19·86354166 &c. *l*.
3. Reduce 15*s* 6*d* to the decimal of a *l*. Ans. ·775*l*.
4. Reduce 7$\frac{1}{2}$*d* to the decimal of a shilling. Ans. ·625*s*.
5. Reduce 5 oz 12 dwts 16 gr to lb. Ans. ·46944 &c. lb.

RULE OF THREE IN DECIMALS.

PREPARE the terms, by reducing the vulgar fractions to decimals, and any compound numbers either to decimals of the higher denominations, or to integers of the lower, also the first and third terms to the same name: Then multiply and divide as in whole numbers.

Note, Any of the convenient Examples in the Rule of Three or Rule of Five in Integers, or Vulgar Fractions, may be taken as proper examples to the same rules in Decimals. —The following Example, which is the first in Vulgar Fractions, is wrought out here, to show the method.

If $\frac{3}{8}$ of a yard of velvet cost $\frac{2}{5}$*l*, what will $\frac{5}{16}$ yd cost?

	yd	*l*	yd	*l*	*s*	*d*
$\frac{3}{8}$ = ·375	·375 :	·4 ::	·3125 :	·333 &c. or	6	8

	·3125	
	·4	
$\frac{2}{5}$ = ·4	·375) ·12500	(·333333 &c.
	1250	20
	125	*s* 6·66666 &c.
$\frac{5}{16}$ = ·3125		12
	Ans. 6*s* 8*d*.	*d* 7·99999 &c. = 8*d*.

DUODECIMALS.

DUODECIMALS, or CROSS MULTIPLICATION, is a rule used by workmen and artificers, in computing the contents of their works.

Dimensions are usually taken in feet, inches, and quarters; any parts smaller than these being neglected as of no consequence. And the same in multiplying them together, or casting up the contents. The method is as follows.

SET down the two dimensions to be multiplied together, one under the other, so that feet may stand under feet, inches under inches, &c.

Multiply each term in the multiplicand, beginning at the lowest, by the feet in the multiplier, and set the result of each straight under its corresponding term, observing to carry 1 for every 12, from the inches to the feet.

In like manner, multiply all the multiplicand by the inches and parts of the multiplier, and set the result of each term one place removed to the right-hand of those in the multiplicand; omitting, however, what is below parts of inches, only carrying to these the proper number of units from the lowest denomination.

Or, instead of multiplying by the inches, take such parts of the multiplicand as there are of a foot.

Then add the two lines together, after the manner of Compound Addition, carrying 1 to the feet for 12 inches, when these come to so many.

EXAMPLES.

1. Multiply	4 f	7 inc	2. Multiply	14 f	9 inc.
by	6	4	by	4	6
	27	6		59	0
	1	$6\frac{1}{3}$		7	$4\frac{1}{2}$
Ans.	29	$0\frac{1}{3}$	Ans.	66	$4\frac{1}{2}$

3. Multiply 4 feet 7 inches by 9 f 6 inc. Ans. 43 f. $6\frac{1}{2}$ inc.
4. Multiply 12 f 5 inc by 6 f 8 inc. Ans. 82 $9\frac{1}{3}$
5. Multiply 35 f $4\frac{1}{2}$ inc by 12 f 3 inc. Ans. 433 $4\frac{1}{8}$
6. Multiply 64 f 6 inc by 8 f $9\frac{1}{4}$ inc. Ans. 565 $8\frac{5}{8}$

INVOLUTION.

Involution is the raising of Powers from any given number, as a root.

A Power is a quantity produced by multiplying any given number, called the Root, a certain number of times continually by itself. Thus,

$$2 = 2 \text{ is the root, or 1st power of } 2.$$
$$2 \times 2 = 4 \text{ is the 2d power, or square of } 2.$$
$$2 \times 2 \times 2 = 8 \text{ is the 3d power, or cube of } 2.$$
$$2 \times 2 \times 2 \times 2 = 16 \text{ is the 4th power of } 2, \&c.$$

And in this manner may be calculated the following Table of the first nine powers of the first 9 numbers.

TABLE of the first Nine Powers of Numbers.

1st	2d	3d	4th	5th	6th	7th	8th	9th
1	1	1	1	1	1	1	1	1
2	4	8	16	32	64	128	256	512
3	9	27	81	243	729	2187	6561	19683
4	16	64	256	1024	4096	16384	65536	262144
5	25	125	625	3125	15625	78125	390625	1953125
6	36	216	1296	7776	46656	279936	1679616	10077696
7	49	343	2401	16807	117649	823543	5764801	40353607
8	64	512	4096	32768	262144	2097152	16777216	134217728
9	81	729	6561	59049	531441	4782969	43046721	387420489

The

The Index or Exponent of a Power, is the number denoting the height or degree of that power; and it is 1 more than the number of multiplications used in producing the same. So 1 is the index or exponent of the 1st power or root, 2 of the 2d power or square, 3 of the third power or cube, 4 of the 4th power, and so on.

Powers, that are to be raised, are usually denoted by placing the index above the root or first power.

So $2^2 = 4$ is the 2d power of 2.
$2^3 = 8$ is the 3d power of 2.
$2^4 = 16$ is the 4th power of 2.
540^4 is the 4th power of 540, &c.

When two or more powers are multiplied together, their product is that power whose index is the sum of the exponents of the factors or powers multiplied. Or the multiplication of the powers, answers to the addition of the indices. Thus, in the following powers of 2,

	1st	2d	3d	4th	5th	6th	7th	8th	9th	10th
	2	4	8	16	32	64	128	256	512	1024
or	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}

Here, $4 \times 4 = 16$, and $2 + 2 = 4$ its index;
and $8 \times 16 = 128$, and $3 + 4 = 7$ its index;
also $16 \times 64 = 1024$, and $4 + 6 = 10$ its index.

OTHER EXAMPLES.

1. What is the 2d power of 45? Ans. 2025.
2. What is the square of 4·16? Ans. 17·3056.
3. What is the 3d power of 3·5? Ans. 42·875.
4. What is the 5th power of ·029? Ans. ·000000020511149.
5. What is the square of $\frac{2}{3}$? Ans. $\frac{4}{9}$.
6. What is the 3d power of $\frac{5}{9}$? Ans. $\frac{125}{729}$.
7. What is the 4th power of $\frac{3}{4}$? Ans. $\frac{81}{256}$.

EVOLUTION.

EVOLUTION, or the reverse of Involution, is the extracting or finding the roots of any given powers.

The root of any number, or power, is such a number, as being multiplied into itself a certain number of times, will produce that power. Thus, 2 is the square root or 2d root of 4, because $2^2 = 2 \times 2 = 4$; and 3 is the cube root or 3d root of 27, because $3^3 = 3 \times 3 \times 3 = 27$.

Any power of a given number or root may be found exactly, namely, by multiplying the number continually into itself. But there are many numbers of which a proposed root can never be exactly found. Yet, by means of decimals, we may approximate or approach towards the root, to any degree of exactness.

Those roots which only approximate, are called Surd roots; but those which can be found quite exact, are called Rational Roots. Thus, the square root of 3 is a surd root; but the square root of 4 is a rational root, being equal to 2: also the cube root of 8 is rational, being equal to 2; but the cube root of 9 is surd or irrational.

Roots are sometimes denoted by writing the character $\surd$ before the power, with the index of the root against it. Thus, the 3d root of 20 is expressed by $\sqrt[3]{20}$; and the square root or 2d root of it is $\sqrt{20}$, the index 2 being always omitted, when only the square root is designed.

When the power is expressed by several numbers, with the sign + or − between them, a line is drawn from the top of the sign over all the parts of it: thus the third root of $45 - 12$ is $\sqrt[3]{45 - 12}$, or thus $\sqrt[3]{}(45 - 12)$, inclosing the numbers in parentheses.

But all roots are now often designed like powers, with fractional indices: thus, the square root of 8 is $8^{\frac{1}{2}}$, the cube root of 25 is $25^{\frac{1}{3}}$, and the 4th root of $45 - 18$ is $\overline{45 - 18}|^{\frac{1}{4}}$, or $(45 - 18)^{\frac{1}{4}}$.

TO

TO EXTRACT THE SQUARE ROOT.

* DIVIDE the given number into periods of two figures each, by setting a point over the place of units, another over the place of hundreds, and so on, over every second figure, both to the left hand in integers, and to the right in decimals.

Find the greatest square in the first period on the left-hand, and set its root on the right-hand of the given number, after the manner of a quotient figure in Division.

---

* The reason for separating the figures of the dividend into periods or portions of two places each, is, that the square of any single figure never consists of more than two places; the square of a number of two figures, of not more than four places, and so on. So that there will be as many figures in the root as the given number contains periods so divided or parted off.

And the reason of the several steps in the operation appears from the algebraic form of the square of any number of terms, whether two or three or more. Thus,

$(a+b)^2 = a^2 + 2ab + b^2 = a^2 + (2a+b)\,b$, the square of two terms; where it appears that a is the first term of the root, and b the second term; also a the first divisor, and the new divisor is $2a+b$, or double the first term increased by the second. And hence the manner of extraction is thus:

$$\begin{array}{r|l}
\text{1st divisor } a\ ) & a^2 + 2ab + b^2\ (\ a+b \text{ the root.} \\
 & a^2 \\
\hline
\text{2d divisor } 2a+b & 2ab + b^2 \\
b & 2ab + b^2 \\
\end{array}$$

Again, for a root of three parts, a, b, c, thus:

$$(a+b+c)^2 = a^2 + 2ab + b^2 + 2ac + 2bc + c^2 =$$
$$a^2 + (2a+b)\,b + (2a+2b+c)\,c,$$ the square of three terms, where a is the first term of the root, b the second, and c the third term; also a the first divisor, $2a+b$ the second, and $2a+2b+c$ the third, each consisting of the double of the root increased by the next term of the same. And the mode of extraction is thus:

$$\begin{array}{r|l}
\text{1st divisor } a\,) & a^2 + 2ab + b^2 + 2ac + 2bc + c^2\ (a+b+c \text{ the root.} \\
 & a^2 \\
\hline
\text{2d divisor } 2a+b & 2ab + b^2 \\
b & 2ab + b^2 \\
\hline
\text{3d divisor } 2a+2b+c & 2ac + 2bc + c^2 \\
c & 2ac + 2bc + c^2 \\
\end{array}$$

Subtract the square thus found from the said period, and to the remainder annex the two figures of the next following period, for a dividend.

Double the root above mentioned for a divisor; and find how often it is contained in the said dividend, exclusive of its right-hand figure; and set that quotient figure both in the quotient and divisor.

Multiply the whole augmented divisor by this last quotient figure, and subtract the product from the said dividend, bringing down to it the next period of the given number, for a new dividend.

Repeat the same process over again, viz. find another new divisor, by doubling all the figures now found in the root; from which, and the last dividend, find the next figure of the root as before; and so on through all the periods, to the last.

Note, The best way of doubling the root, to form the new divisors, is by adding the last figure always to the last divisor, as appears in the following examples.—Also, after the figures belonging to the given number are all exhausted, the operation may be continued into decimals at pleasure, by adding any number of periods of ciphers, two in each period.

EXAMPLES.

1. To find the square root of 29506624.

29506624 (5432 the root.
25

104	450
4	416
1083	3466
3	3249
10862	21724
2	21724

NOTE, *When the root is to be extracted to many places of figures, the work may be considerably shortened, thus:*

Having proceeded in the extraction after the common method, till there be found half the required number of figures in

in the root, or one figure more; then, for the rest, divide the last remainder by its corresponding divisor, after the manner of the third contraction in Division of Decimals; thus,

2. To find the root of 2 to nine places of figures.

2 (1·41421356 the root.
1

24	100
4	96
281	400
1	281
2824	11900
4	11296
28282	60400
2	56564

28284) 3836 (1356
···· 1008
160
19
2

3. What is the square root of 2025?	Ans. 45.
4. What is the square root of 17·3056?	Ans. 4·16.
5. What is the square root of ·000729?	Ans. ·027.
6. What is the square root of 3?	Ans. 1·732050.
7. What is the square root of 5?	Ans. 2·236068.
8. What is the square root of 6?	Ans. 2·449489.
9. What is the square root of 7?	Ans. 2·645751.
10. What is the square root of 10?	Ans. 3·162277.
11. What is the square root of 11?	Ans. 3·316624.
12. What is the square root of 12?	Ans. 3·464101.

RULES FOR THE SQUARE ROOTS OF VULGAR FRACTIONS AND MIXED NUMBERS.

First prepare all vulgar fractions, by reducing them to their least terms, both for this and all other roots. Then

1. Take the root of the numerator and of the denominator for the respective terms of the root required. And this is the best way if the denominator be a complete power: but if it be not, then

2. Multiply the numerator and denominator together; take the root of the product: this root being made the nume-

 rator

rator to the denominator of the given fraction, or made the denominator to the numerator of it, will form the fractional root required.

$$\text{That is, } \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{ab}}{b} = \frac{a}{\sqrt{ab}}.$$

And this rule will serve, whether the root be finite or infinite.

3. Or reduce the vulgar fraction to a decimal, and extract its root.

4. Mixed numbers may be either reduced to improper fractions, and extracted by the first or second rule, or the vulgar fraction may be reduced to a decimal, then joined to the integer, and the root of the whole extracted.

EXAMPLES.

1. What is the root of $\frac{25}{36}$? Ans. $\frac{5}{6}$.
2. What is the root of $\frac{27}{147}$? Ans. $\frac{3}{7}$.
3. What is the root of $\frac{9}{12}$? Ans. 0·866025.
4. What is the root of $\frac{5}{12}$? Ans. 0·645497.
5. What is the root of $17\frac{3}{8}$? Ans. 4·168333.

By means of the square root also may readily be found the 4th root, or the 8th root, or the 16th root, &c, that is, the root of any power whose index is some power of the number 2; namely, by extracting so often the square root as is denoted by that power of 2; that is, two extractions for the 4th root, three for the 8th root, and so on.

So, to find the 4th root of the number 21035·8, extract the square root two times as follows:

```
        21035·8000        (145·037237 (12·0431407 the 4th root.
        1                  1

 24 |  110             22 | 45
  4 |   96              2 | 44

285 |  1435          2404 | 10372
  5 |  1425             4 |  9616

29003 | 108000 24083      |  75637
    3 |  87009     3      |  72249

          20991 (7237        3388 ( 1407
            687               980
            107                17
```

Ex. 2. What is the 4th root of 97·41?

TO

TO EXTRACT THE CUBE ROOT.

I. *By the Common Rule**.

1. HAVING divided the given number into periods of three figures each, (by setting a point over the place of units, and also over every third figure, from thence, to the left hand in whole numbers, and to the right in decimals), find the nearest less cube to the first period; set its root in the quotient, and subtract the said cube from the first period; to the remainder bring down the second period, and call this the resolvend.

2. To three times the square of the root, just found, add three times the root itself, setting this one place more to the right than the former, and call this sum the divisor. Then divide the resolvend, wanting the last figure, by the divisor, for the next figure of the root, which annex to the former; calling this last figure e, and the part of the root before found let be called a.

3. Add all together these three products, namely, thrice a square multiplied by e, thrice a multiplied by e square, and e cube, setting each of them one place more to the right than the former, and call the sum the subtrahend; which must not exceed the resolvend; but if it does, then make the last figure e less, and repeat the operation for finding the subtrahend, till it be less than the resolvend.

4. From the resolvend take the subtrahend, and to the remainder join the next period of the given number for a new resolvend; to which form a new divisor from the whole root now found; and from thence another figure of the root, as directed in Article 2, and so on.

---

* The reason for pointing the given number into periods of three figures each, is because the cube of one figure never amounts to more than three places. And, for a similar reason, a given number is pointed into periods of four figures for the 4th root, of five figures for the 5th root, and so on.

And the reason for the other parts of the rule depends on the algebraic formation of a cube: for, if the root consist of the two parts $a + b$, then its cube is as follows: $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$; where a is the root of the first part a^3; the resolvend is $3a^2b + 3ab^2 + b^3$, which is also the same as the three parts of the subtrahend; also the divisor is $3a^2 + 3a$, by which dividing the first two terms of the resolvend $3a^2b + ab^2$, gives b for the second part of the root; and so on.

EXAMPLE.

EXAMPLE.

To extract the cube root of 48228·544.

$$
\begin{array}{r|l}
3 \times 3^2 = 27 & 48228{\cdot}544 \ (\ 36{\cdot}4 \text{ root.} \\
3 \times 3 = 09 & 27 \\
\hline
\text{Divisor } 279 & 21228 \text{ resolvend.}
\end{array}
$$

$$
\left.\begin{array}{r}
3 \times 3^2 \times 6 = 162 \\
3 \times 3 \times 6^2 = 324 \\
6^3 = 216
\end{array}\right\} \text{add}
$$

$$
\begin{array}{r|l}
3 \times 36^2 = 3888 & 19656 \text{ subtrahend.} \\
3 \times 36 = 108 & \\
\hline
38988 & 1572544 \text{ resolvend.}
\end{array}
$$

$$
\left.\begin{array}{r}
3 \times 36^2 \times 4 = 15552 \\
3 \times 36 \times 4^2 = 1728 \\
4^3 = 64
\end{array}\right\} \text{add}
$$

$$
\begin{array}{l}
1572544 \text{ subtrahend.} \\
\hline
0000000 \text{ remainder.}
\end{array}
$$

Ex. 2. Extract the cube root of 571482·19.
Ex. 3. Extract the cube root of 1628·1582.
Ex. 4. Extract the cube root of 1332.

II. *To extract the Cube Root by a short Way**.

1. By trials, or by the table of roots at p. 90, &c, take the nearest rational cube to the given number, whether it be greater or less; and call it the assumed cube.

2. Then

* The method usually given for extracting the cube root, is so exceedingly tedious, and difficult to be remembered, that various other approximating rules have been invented, viz. by Newton, Raphson, Halley, De Lagny, Simpson, Emerson, and several other mathematicians; but no one that I have yet seen, is so simple in its form, or seems so well adapted for general use, as that above given. This rule is the same in effect as Dr. Halley's rational formula,

2. Then say, by the Rule of Three, As the sum of the given number and double the assumed cube, is to the sum of the assumed cube and double the given number, so is the root of the assumed cube, to the root required, nearly. Or, As the first sum is to the difference of the given and assumed cube, so is the assumed root to the difference of the roots nearly.

3. Again, by using, in like manner, the cube of the root last found as a new assumed cube, another root will be obtained still nearer. And so on as far as we please; using always the cube of the last found root, for the assumed cube.

EXAMPLE.

To find the cube root of 21035·8.

Here we soon find that the root lies between 20 and 30, and then between 27 and 28. Taking therefore 27, its cube is 19683, which is the assumed cube. Then

	19683		21035·8				
	2		2				
	39366		42071·6				
	21035·8		19683				
As	60401·8	:	61754·6	::	27	:	27·6047.
			27				
			4322822				
			1235092				

60401·8) 1667374·2 (27·6047 the root nearly.
459338
36525
284
42

formula, but more commodiously expressed; and the first investigation of it was given in my Tracts, p. 49. The algebraic form of it is this:

As $P + 2A : A + 2P :: r : R$. Or,
As $P + 2A : P \backsim A :: r : R \backsim r$;

where P is the given number, A the assumed nearest cube, r the cube root of A, and R the root of P sought.

Again,

Again, for a second operation, the cube of this root is 21035·318645155823, and the process by the latter method will be thus:

21035·318645, &c.	
42070·637290	21035·8
21035·8	21035·318645, &c.
As 63106·43729 :	diff. ·481355 :: 27·6047 :
	the diff. ·000210560

conseq. the root req. is 27·604910560.

Ex. 2. To extract the cube root of ·67.

Ex. 3. To extract the cube root of ·01.

TO EXTRACT ANY ROOT WHATEVER*.

LET P be the given power or number, n the index of the power, A the assumed power, r its root, R the required root of P. Then say,

As the sum of $n+1$ times A and $n-1$ times P, is to the sum of $n+1$ times P and $n-1$ times A; so is the assumed root r, to the required root R.

Or, as half the said sum of $n+1$ times A, and $n-1$ times P, is to the difference between the given and assumed powers, so is the assumed root r, to the difference between the true and assumed roots; which difference, added or subtracted, as the case requires, gives the true root nearly.

That is, $\overline{n+1}.\,A+\overline{n-1}.\,P : \overline{n+1}.\,P.+\overline{n-1}.\,A :: r : R$.

Or, $\overline{n+1}.\,\tfrac{1}{2}A+\overline{n-1}.\,\tfrac{1}{2}P : P \backsim A :: r : R \backsim r$.

And the operation may be repeated as often as we please, by using always the last found root for the assumed root, and its nth power for the assumed power A.

* This is a very general approximating rule, of which that for the cube root is a particular case, and is the best adapted for practice, and for memory, of any that I have yet seen. It was first discovered in this form by myself, and the investigation and use of it were given at large in my Tracts, p. 45, &c.

EXAMPLE.

EXAMPLE.

To extract the 5th root of 21035·8.

Here it appears that the 5th root is between 7·3 and 7·4. Taking 7·3, its 5th power is 20730·71593. Hence we have P = 21035·8, n = 5, r = 7·3 and A = 20730·71593; then

$\overline{n+1}.\frac{1}{2}A + \overline{n-1}.\frac{1}{2}P : P \backsim A :: r : R \backsim r$, that is,

3 × 20730·71593 + 2 × 21035·8 : 305·084 :: 7·3 :

3	2	7·3	
62192·14779	42071·6	915252	
42071·6		2135588	
104263·74779		2227·1132	(·0213605 = R ∽ r
			7·3 = r, add.
			7·321360 = R, true to the last figure.

OTHER EXAMPLES.

1. What is the 3d root of 2?	Ans. 1·259921.
2. What is the 3d root of 3214?	Ans. 14·75758.
3. What is the 4th root of 2?	Ans. 1·189207.
4. What is the 4th root of 97·41?	Ans. 3·1415999.
5. What is the 5th root of 2?	Ans. 1·148699.
6. What is the 6th root of 21035·8?	Ans. 5·254037.
7. What is the 6th root of 2?	Ans. 1·122462.
8. What is the 7th root of 21035·8?	Ans. 4·145392.
9. What is the 7th root of 2?	Ans. 1·104089.
10. What is the 8th root of 21035·8?	Ans. 3·470323.
11. What is the 8th root of 2?	Ans. 1·090508.
12. What is the 9th root of 21035·8?	Ans. 3·022239.
13. What is the 9th root of 2?	Ans. 1·080059.

The following is a Table of squares and cubes, as also the square roots and cube roots, of all numbers from 1 to 1000, which will be found very useful on many occasions, in numeral calculations, when roots or powers are concerned.

A TABLE

Number.	Square.	Cube.	Square Root.	Cube Root.
1	1	1	1·0000000	1·000000
2	4	8	1·4142136	1·259921
3	9	27	1·7320508	1·442250
4	16	64	2·0000000	1·587401
5	25	125	2·2360680	1·709976
6	36	216	2·4494897	1·817121
7	49	343	2·6457513	1·912933
8	64	512	2·8284271	2·000000
9	81	729	3·0000000	2·080084
10	100	1000	3·1622777	2·154435
11	121	1331	3·3166248	2·223980
12	144	1728	3·4641016	2·289428
13	169	2197	3·6055513	2·351335
14	196	2744	3·7416574	2·410142
15	225	3375	3·8729833	2·466212
16	256	4096	4·0000000	2·519842
17	289	4913	4·1231056	2·571282
18	324	5832	4·2426407	2·620741
19	361	6859	4·3588989	2·668402
20	400	8000	4·4721360	2·714418
21	441	9261	4·5825757	2·758923
22	484	10648	4·6904158	2·802039
23	529	12167	4·7958315	2·843867
24	576	13824	4·8989795	2·884499
25	625	15625	5·0000000	2·924018
26	676	17576	5·0990195	2·962496
27	729	19683	5·1961524	3·000000
28	784	21952	5·2915026	3·036589
29	841	24389	5·3851648	3·072317
30	900	27000	5·4772256	3·107232
31	961	29791	5·5677644	3·141381
32	1024	32768	5·6568542	3·174802
33	1089	35937	5·7445626	3·207534
34	1156	39304	5·8309519	3·239612
35	1225	42875	5·9160798	3·271066
36	1296	46656	6·0000000	3·301927
37	1369	50653	6·0827625	3·332222
38	1444	54872	6·1644140	3·361975
39	1521	59319	6·2449980	3·391211
40	1600	64000	6·3245553	3·419952
41	1681	68921	6·4031242	3·448217
42	1764	74088	6·4807407	3·476027
43	1849	79507	6·5574385	3·503398
44	1936	85184	6·6332496	3·530348
45	2025	91125	6·7082039	3·556893
46	2116	97336	6·7823300	3·583048
47	2209	103823	6·8556546	3·608826
48	2304	110592	6·9282032	3·634241
49	2401	117649	7·0000000	3·659306
50	2500	125000	7·0710678	3·684031

Number.	Square.	Cube.	Square Root.	Cube Root.
51	2601	132651	7·1414284	3·708430
52	2704	140608	7·2111026	3·732511
53	2809	148877	7·2801099	3·756286
54	2916	157464	7·3484692	3·779763
55	3025	166375	7·4161985	3·802953
56	3136	175616	7·4833148	3·825862
57	3249	185193	7·5498344	3·848501
58	3364	195112	7·6157731	3·870877
59	3481	205379	7·6811457	3·892996
60	3600	216000	7·7459667	3·914867
61	3721	226981	7·8102497	3·936497
62	3844	238328	7·8740079	3·957892
63	3969	250047	7·9372539	3·979057
64	4096	262144	8·0000000	4·000000
65	4225	274625	8·0622577	4·020726
66	4356	287496	8·1240384	4·041240
67	4489	300763	8·1853528	4·061548
68	4624	314432	8·2462113	4·081656
69	4761	328509	8·3066239	4·101566
70	4900	343000	8·3666003	4·121285
71	5041	357911	8·4261498	4·140818
72	5184	373248	8·4852814	4·160168
73	5329	389017	8·5440037	4·179339
74	5476	405224	8·6023253	4·198336
75	5625	421875	8·6602540	4·217163
76	5776	438976	8·7177979	4·235824
77	5929	456533	8·7749644	4·254321
78	6084	474552	8·8317609	4·272659
79	6241	493039	8·8881944	4·290841
80	6400	512000	8·9442719	4·308870
81	6561	531441	9·0000000	4·326749
82	6724	551368	9·0553851	4·344481
83	6889	571787	9·1104336	4·362071
84	7056	592704	9·1651514	4·379519
85	7225	614125	9·2195445	4·396830
86	7396	636056	9·2736185	4·414005
87	7569	658503	9·3273791	4·431047
88	7744	681472	9·3808315	4·447960
89	7921	704969	9·4339811	4·464745
90	8100	729000	9·4868330	4·481405
91	8281	753571	9·5393920	4·497942
92	8464	778688	9·5916630	4·514357
93	8649	804357	9·6436508	4·530655
94	8836	830584	9·6953597	4·546836
95	9025	857375	9·7467943	4·562903
96	9216	884736	9·7979590	4·578857
97	9409	912673	9·8488578	4·594701
98	9604	941192	9·8994949	4·610436
99	9801	970299	9·9498744	4·626065
100	10000	1000000	10·0000000	4·641589

Number.	Square.	Cube.	Square Root.	Cube Root.
101	10201	1030301	10·0498756	4·657010
102	10404	1061208	10·0995049	4·672330
103	10609	1092727	10·1488916	4·687548
104	10816	1124864	10·1980390	4·702669
105	11025	1157625	10·2469508	4·717694
106	11236	1191016	10·2956301	4·732624
107	11449	1225043	10·3440804	4·747459
108	11664	1259712	10·3923048	4·762203
109	11881	1295029	10·4403065	4·776856
110	12100	1331000	10·4880885	4·791420
111	12321	1367631	10·5356538	4·805896
112	12544	1404928	10·5830052	4·820284
113	12769	1442897	10·6301458	4·834588
114	12996	1481544	10·6770783	4·848808
115	13225	1520875	10·7238053	4·862944
116	13456	1560896	10·7703296	4·876999
117	13689	1601613	10·8166538	4·890973
118	13924	1643032	10·8627805	4·904868
119	14161	1685159	10·9087121	4·918685
120	14400	1728000	10·9544512	4·932424
121	14641	1771561	11·0000000	4·946088
122	14884	1815848	11·0453610	4·959675
123	15129	1860867	11·0905365	4·973190
124	15376	1906624	11·1355287	4·986631
125	15625	1953125	11·1803399	5·000000
126	15876	2000376	11·2249722	5·013298
127	16129	2048383	11·2694277	5·026526
128	16384	2097152	11·3137085	5·039684
129	16641	2146689	11·3578167	5·052774
130	16900	2197000	11·4017543	5·065797
131	17161	2248091	11·4455231	5·078753
132	17424	2299968	11·4891253	5·091643
133	17689	2352637	11·5325626	5·104469
134	17956	2406104	11·5758369	5·117230
135	18225	2460375	11·6189500	5·129928
136	18496	2515456	11·6619038	5·142563
137	18769	2571353	11·7046999	5·155137
138	19044	2628072	11·7473444	5·167649
139	19321	2685619	11·7898261	5·180101
140	19600	2744000	11·8321596	5·192494
141	19881	2803221	11·8743421	5·204828
142	20164	2863288	11·9163753	5·217103
143	20449	2924207	11·9582607	5·229321
144	20736	2985984	12·0000000	5·241482
145	21025	3048625	12·0415946	5·253588
146	21316	3112136	12·0830460	5·265637
147	21609	3176523	12·1243557	5·277632
148	21904	3241792	12·1655251	5·289572
149	22201	3307949	12·2065556	5·301459
150	22500	3375000	12·2474487	5·313293

Number.	Square.	Cube.	Square Root.	Cube Root.
151	22801	3442951	12·2882057	5·325074
152	23104	3511808	12·3288280	5·336803
153	23409	3581577	12·3693169	5·348481
154	23716	3652264	12·4096736	5·360108
155	24025	3723875	12·4498996	5·371685
156	24336	3796416	12·4899960	5·383213
157	24649	3869893	12·5299641	5·394690
158	24964	3944312	12·5698051	5·406120
159	25281	4019679	12·6095202	5·417501
160	25600	4096000	12·6491106	5·428835
161	25921	4173281	12·6885775	5·440122
162	26244	4251528	12·7279221	5·451362
163	26569	4330747	12·7671453	5·462556
164	26896	4410944	12·8062485	5·473703
165	27225	4492125	12·8452326	5·484806
166	27556	4574296	12·8840987	5·495865
167	27889	4657463	12·9228480	5·506879
168	28224	4741632	12·9614814	5·517848
169	28561	4826809	13·0000000	5·528775
170	28900	4913000	13·0384048	5·539658
171	29241	5000211	13·0766968	5·550499
172	29584	5088448	13·1148770	5·561298
173	29929	5177717	13·1529464	5·572054
174	30276	5268024	13·1909060	5·582770
175	30625	5359375	13·2287566	5·593445
176	30976	5451776	13·2664992	5·604079
177	31329	5545233	13·3041347	5·614673
178	31684	5639752	13·3416641	5·625226
179	32041	5735339	13·3790882	5·635741
180	32400	5832000	13·4164079	5·646216
181	32761	5929741	13·4536240	5·656652
182	33124	6028568	13·4907376	5·667051
183	33489	6128487	13·5277493	5·677411
184	33856	6229504	13·5646600	5·687734
185	34225	6331625	13·6014705	5·698019
186	34596	6434856	13·6381817	5·708267
187	34969	6539203	13·6747943	5·718479
188	35344	6644672	13·7113092	5·728654
189	35721	6751269	13·7477271	5·738794
190	36100	6859000	13·7840488	5·748897
191	36481	6967871	13·8202750	5·758965
192	36864	7077888	13·8564065	5·768998
193	37249	7189057	13·8924440	5·778996
194	37636	7301384	13·9283883	5·788960
195	38025	7414875	13·9642400	5·798890
196	38416	7529536	13·0000000	5·808786
197	38809	7645373	14·0356688	5·818648
198	39204	7762392	14·0712473	5·828476
199	39601	7880599	14·1067360	5·838272
200	40000	8000000	14·1421356	5·848035

Number.	Square.	Cube.	Square Root.	Cube Root.
101	10201	1030301	10·0498756	4·657010
102	10404	1061208	10·0995049	4·672330
103	10609	1092727	10·1488916	4·687548
104	10816	1124864	10·1980390	4·702669
105	11025	1157625	10·2469508	4·717694
106	11236	1191016	10·2956301	4·732624
107	11449	1225043	10·3440804	4·747459
108	11664	1259712	10·3923048	4·762203
109	11881	1295029	10·4403065	4·776856
110	12100	1331000	10·4880885	4·791420
111	12321	1367631	10·5356538	4·805896
112	12544	1404928	10·5830052	4·820284
113	12769	1442897	10·6301458	4·834588
114	12996	1481544	10·6770783	4·848808
115	13225	1520875	10·7238053	4·862944
116	13456	1560896	10·7703296	4·876999
117	13689	1601613	10·8166538	4·890973
118	13924	1643032	10·8627805	4·904868
119	14161	1685159	10·9087121	4·918685
120	14400	1728000	10·9544512	4·932424
121	14641	1771561	11·0000000	4·946088
122	14884	1815848	11·0453610	4·959675
123	15129	1860867	11·0905365	4·973190
124	15376	1906624	11·1355287	4·986631
125	15625	1953125	11·1803399	5·000000
126	15876	2000376	11·2249722	5·013298
127	16129	2048383	11·2694277	5·026526
128	16384	2097152	11·3137085	5·039684
129	16641	2146689	11·3578167	5·052774
130	16900	2197000	11·4017543	5·065797
131	17161	2248091	11·4455231	5·078753
132	17424	2299968	11·4891253	5·091643
133	17689	2352637	11·5325626	5·104469
134	17956	2406104	11·5758369	5·117230
135	18225	2460375	11·6189500	5·129928
136	18496	2515456	11·6619038	5·142563
137	18769	2571353	11·7046999	5·155137
138	19044	2628072	11·7473444	5·167649
139	19321	2685619	11·7898261	5·180101
140	19600	2744000	11·8321596	5·192494
141	19881	2803221	11·8743421	5·204828
142	20164	2863288	11·9163753	5·217103
143	20449	2924207	11·9582607	5·229321
144	20736	2985984	12·0000000	5·241482
145	21025	3048625	12·0415946	5·253588
146	21316	3112136	12·0830460	5·265637
147	21609	3176523	12·1243557	5·277632
148	21904	3241792	12·1655251	5·289572
149	*22201*	3307949	12·2065556	5·301459
150	*22500*	3375000	12·2474487	5·313293

Numb.	Square.	Cube.	Square Root.	Cube Root
251	63001	15813251	15·8429795	6·307992
252	63504	16003008	15·8745079	6·316359
253	64009	16194277	15·9059737	6·324704
254	64516	16387064	15·9373775	6·333025
255	65025	16581375	15·9687194	6·341325
256	65536	16777216	16·0000000	6·349602
257	66049	16974593	16·0312195	6·357859
258	66564	17173512	16·0623784	6·366095
259	67081	17373979	16·0934769	6·374310
260	67600	17576000	16·1245155	6·382504
261	68121	17779581	16·1554944	6·390676
262	68644	17984728	16·1864141	6·398827
263	69169	18191447	16·2172747	6·406958
264	69696	18399744	16·2480768	6·415068
265	70225	18609625	16·2788206	6·423157
266	70756	18821096	16·3095064	6·431226
267	71289	19034163	16·3401346	6·439275
268	71824	19248832	16·3707055	6·447305
269	72361	19465109	16·4012195	6·455314
270	72900	19683000	16·4316767	6·463304
271	73441	19902511	16·4620776	6·471274
272	73984	20123648	16·4924225	6·479224
273	74529	20346417	16·5227116	6·487153
274	75076	20570824	16·5529454	6·495064
275	75625	20796875	16·5831240	6·502956
276	76176	21024576	16·6132477	6·510829
277	76729	21253933	16·6433170	6·518684
278	77284	21484952	16·6733320	6·526519
279	77841	21717639	16·7032931	6·534335
280	78400	21952000	16·7332005	6·542132
281	78961	22188041	16·7630546	6·549911
282	79524	22425768	16·7928556	6·557672
283	80089	22665187	16·8226038	6·565415
284	80656	22906304	16·8522995	6·573139
285	81225	23149125	16·8819430	6·580844
286	81796	23393656	16·9115345	6·588531
287	82369	23639903	16·9410743	6·596202
288	82944	23887872	16·9705627	6·603854
289	83521	24137569	17·0000000	6·611488
290	84100	24389000	17·0293864	6·619106
291	84681	24642171	17·0587221	6·626705
292	85264	24897088	17·0880075	6·634287
293	85849	25153757	17·1172428	6·641851
294	86436	25412184	17·1464282	6·649399
295	87025	25672375	17·1755640	6·656930
296	87616	25934336	17·2046505	6·664443
297	88209	26198073	17·2336879	6·671940
298	88804	26463592	17·2626765	6·679419
299	89401	26730899	17·2916165	6·686882
300	90000	27000000	17·3205081	6·694328

Numb.	Square.	Cube.	Square Root.	Cube Root.
301	90601	27270901	17·3493516	6·701758
302	91204	27543608	17·3781472	6·709172
303	91809	27818127	17·4068952	6·716569
304	92416	28094464	17·4355958	6·723950
305	93025	28372625	17·4642492	6·731316
306	93636	28652616	17·4928557	6·738665
307	94249	28934443	17·5214155	6·745997
308	94864	29218112	17·5499288	6·753313
309	95481	29503629	17·5783958	6·760614
310	96100	29791000	17·6068169	6·767899
311	96721	30080231	17·6351921	6·775168
312	97344	30371328	17·6635217	6·782422
313	97969	30664297	17·6918060	6·789661
314	98596	30959144	17·7200451	6·796884
315	99225	31255875	17·7482393	6·804091
316	99856	31554496	17·7763888	6·811284
317	100489	31855013	17·8044938	6·818461
318	101124	32157432	17·8325545	6·825624
319	101761	32461759	17·8605711	6·832771
320	102400	32768000	17·8885438	6·839903
321	103041	33076161	17·9164729	6·847021
322	103684	33386248	17·9443584	6·854124
323	104329	33698267	17·9722008	6·861211
324	104976	34012224	18·0000000	6·868284
325	105625	34328125	18·0277564	6·875343
326	106276	34645976	18·0554701	6·882388
327	106929	34965783	18·0831413	6·889419
328	107584	35287552	18·1107703	6·896435
329	108241	35611289	18·1383571	6·903436
330	108900	35937000	18·1659021	6·910423
331	109561	36264691	18·1934054	6·917396
332	110224	36594368	18·2208672	6·924355
333	110889	36926037	18·2482876	6·931300
334	111556	37259704	18·2756669	6·938232
335	112225	37595375	18·3030052	6·945149
336	112896	37933056	18·3303028	6·952053
337	113569	38272753	18·3575598	6·958943
338	114244	38614472	18·3847763	6·965819
339	114921	38958219	18·4119526	6·972682
340	115600	39304000	18·4390889	6·979532
341	116281	39651821	18·4661853	6·986369
342	116964	40001688	18·4932420	6·993191
343	117649	40353607	18·5202592	7·000000
344	118336	40707584	18·5472370	7·006796
345	119025	41063625	18·5741756	7·013579
346	119716	41421736	18·6010752	7·020349
347	120409	41781923	18·6279360	7·027106
348	121104	42144192	18·6547581	7·033850
349	121801	42508549	18·6815417	7·040581
350	122500	42875000	18·7082869	7·047208

Numb.	Square.	Cube.	Square Root.	Cube Root.
351	123201	43243551	18·7349940	7·054003
352	123904	43614208	18·7616630	7·060696
353	124609	43986977	18·7882942	7·067376
354	125316	44361864	18·8148877	7·074043
355	126025	44738875	18·8414437	7·080698
356	126736	45118016	18·8679623	7·087341
357	127449	45499293	18·8944436	7·093970
358	128164	45882712	18·9208879	7·100588
359	128881	46268279	18·9472953	7·107193
360	129600	46656000	18·9736660	7·113786
361	130321	47045881	19·0000000	7·120367
362	131044	47437928	19·0262976	7·126935
363	131769	47832147	19·0525589	7·133492
364	132496	48228544	19·0787840	7·140037
365	133225	48627125	19·1049732	7·146569
366	133956	49027896	19·1311265	7·153090
367	134689	49430863	19·1572441	7·159599
368	135424	49836032	19·1833261	7·166095
369	136161	50243409	19·2093727	7·172580
370	136900	50653000	19·2353841	7·179054
371	137641	51064811	19·2613603	7·185516
372	138384	51478848	19·2873015	7·191966
373	139129	51895117	19·3132079	7·198405
374	139876	52313624	19·3390796	7·204832
375	140625	52734375	19·3649167	7·211247
376	141376	53157376	19·3907194	7·217652
377	142129	53582633	19·4164878	7·224045
378	142884	54010152	19·4422221	7·230427
379	143641	54439939	19·4679223	7·236797
380	144400	54872000	19·4935887	7·243156
381	145161	55306341	19·5192213	7·249504
382	145924	55742968	19·5448203	7·255841
383	146689	56181887	19·5703858	7·262167
384	147456	56623104	19·5959179	7·268482
385	148225	57066625	19·6214169	7·274786
386	148996	57512456	19·6468827	7·281079
387	149769	57960603	19·6723156	7·287362
388	150544	58411072	19·6977156	7·293633
389	151321	58863869	19·7230829	7·299893
390	152100	59319000	19·7484177	7·306143
391	152881	59776471	19·7737199	7·312383
392	153664	60236288	19·7989899	7·318611
393	154449	60698457	19·8242276	7·324829
394	155236	61162984	19·8494332	7·331037
395	156025	61629875	19·8746069	7·337234
396	156816	62099136	19·8997487	7·343420
397	157609	62570773	19·9248588	7·349596
398	158404	63044792	19·9499373	7·355762
399	159201	63521199	19·9749844	7·361917
400	160000	64000000	20·0000000	7·368063

Numb.	Square.	Cube.	Square Root.	Cube Root.
401	160801	64481201	20·0249844	7·374198
402	161604	64964808	20·0499377	7·380322
403	162409	65450827	20·0748599	7·386437
404	163216	65939264	20·0997512	7·392542
405	164025	66430125	20·1246118	7·398636
406	164836	66923416	20·1494417	7·404720
407	165649	67419143	20·1742410	7·410794
408	166464	67911312	20·1990099	7·416859
409	167281	68417929	20·2237484	7·422914
410	168100	68921000	20·2484567	7·428958
411	168921	69426531	20·2731349	7·434993
412	169744	69934528	20·2977831	7·441018
413	170569	70444997	20·3224014	7·447033
414	171396	70957944	20·3469899	7·453039
415	172225	71473375	20·3715488	7·459036
416	173056	71991296	20·3960781	7·465022
417	173889	72511713	20·4205779	7·470999
418	174724	73034632	20·4450483	7·476966
419	175561	73560059	20·4694895	7·482924
420	176400	74088000	20·4939015	7·488872
421	177241	74618461	20·5182845	7·494810
422	178084	75151448	20·5426386	7·500740
423	178929	75686967	20·5669638	7·506660
424	179776	76225024	20·5912603	7·512571
425	180625	76765625	20·6155281	7·518473
426	181476	77308776	20·6397674	7·524365
427	182329	77854483	20·6639783	7·530248
428	183184	78402752	20·6881609	7·536121
429	184041	78953589	20·7123152	7·541986
430	184900	79507000	20·7364414	7·547841
431	185761	80062991	20·7605395	7·553688
432	186624	80621568	20·7846097	7·559525
433	187489	81182737	20·8086520	7·565353
434	188356	81746504	20·8326667	7·571173
435	189225	82312875	20·8566536	7·576984
436	190096	82881856	20·8806130	7·582786
437	190969	83453453	20·9045450	7·588579
438	191844	84027672	20·9284495	7·594363
439	192721	84604519	20·9523268	7·600138
440	193600	85184000	20·9761770	7·605905
441	194481	85766121	21·0000000	7·611662
442	195364	86350888	21·0237960	7·617411
443	196249	86938307	21·0475652	7·623151
444	197136	87528384	21·0713075	7·628883
445	198025	88121125	21·0950231	7·634606
446	198916	88716536	21·1187121	7·640321
447	199809	89314623	21·1423745	7·646027
448	200704	89915392	21·1660105	7·651725
449	201601	90518849	21·1896201	7·657414
450	202500	[illegible]25000	21·2132034	7·663094

Numb.	Square.	Cube.	Square Root.	Cube Root
451	203401	91733851	21·2367606	7·668766
452	204304	92345408	21·2602916	7·674430
453	205209	92959677	21·2837967	7·680085
454	206116	93576664	21·3072758	7·685732
455	207025	94196375	21·3307290	7·691371
456	207936	94818816	21·3541565	7·697002
457	208849	95443993	21·3775583	7·702624
458	209764	96071912	21·4009346	7·708238
459	210681	96702579	21·4242853	7·713844
460	211600	97336000	21·4476106	7·719442
461	212521	97972181	21·4709106	7·725032
462	213444	98611128	21·4941853	7·730614
463	214369	99252847	21·5174348	7·736187
464	215296	99897344	21·5406592	7·741753
465	216225	100544625	21·5638587	7·747310
466	217156	101194696	21·5870331	7·752860
467	218089	101847563	21·6101828	7·758402
468	219024	102503232	21·6333077	7·763936
469	219961	103161709	21·6564078	7·769462
470	220900	103823000	21·6794834	7·774980
471	221841	104487111	21·7025344	7·780490
472	222784	105154048	21·7255610	7·785992
473	223729	105823817	21·7485632	7·791487
474	224676	106496424	21·7715411	7·796974
475	225625	107171875	21·7944947	7·802453
476	226576	107850176	21·8174242	7·807925
477	227529	108531333	21·8403297	7·813389
478	228484	109215352	21·8632111	7·818845
479	229441	109902239	21·8860686	7·824294
480	230400	110592000	21·9089023	7·829735
481	231361	111284641	21·9317122	7·835168
482	232324	111980168	21·9544984	7·840594
483	233289	112678587	21·9772610	7·846013
484	234256	113379904	22·0000000	7·851424
485	235225	114084125	22·0227155	7·856828
486	236196	114791256	22·0454077	7·862224
487	237169	115501303	22·0680765	7·867613
488	238144	116214272	22·0907220	7·872994
489	239121	116930169	22·1133444	7·878368
490	240100	117649000	22·1359436	7·883734
491	241081	118370771	22·1585198	7·889094
492	242064	119095488	22·1810730	7·894446
493	243049	119823157	22·2036033	7·899791
494	244036	120553784	22·2261108	7·905129
495	245025	121287375	22·2485955	7·910460
496	246016	122023936	22·2710575	7·915784
497	247009	122763473	22·2934968	7·921100
498	248004	123505992	22·3159136	7·926408
499	249001	124251499	22·3383079	7·931710
500	250000	125000000	22·3606798	7·937005

Numb.	Square.	Cube.	Square Root.	Cube Root.
501	251001	125751501	22·3830293	7·942293
502	252004	126506008	22·4053565	7·947573
503	253009	127263527	22·4276615	7·952847
504	254016	128024064	22·4499443	7·958114
505	255025	128787625	22·4722051	7·963374
506	256036	129554216	22·4944438	7·968627
507	257049	130323843	22·5166605	7·973873
508	258064	131096512	22·5388553	7·979112
509	259081	131872229	22·5610283	7·984344
510	260100	132651000	22·5831796	7·989569
511	261121	133432831	22·6053091	7·994788
512	262144	134217728	22·6274170	8·000000
513	263169	135005697	22·6495033	8·005205
514	264196	135796744	22·6715681	8·010403
515	265225	136590875	22·6936114	8·015595
516	266256	137388096	22·7156334	8·020779
517	267289	138188413	22·7376340	8·025957
518	268324	138991832	22·7596134	8·031129
519	269361	139798359	22·7815715	8·036293
520	270400	140608000	22·8035085	8·041451
521	271441	141420761	22·8254244	8·046603
522	272484	142236648	22·8473193	8·051748
523	273529	143055667	22·8691933	8·056886
524	274576	143877824	22·8910463	8·062018
525	275625	144703125	22·9128785	8·067143
526	276676	145531576	22·9346899	8·072262
527	277729	146363183	22·9564806	8·077374
528	278784	147197952	22·9782506	8·082480
529	279841	148035889	23·0000000	8·087579
530	280900	148877000	23·0217289	8·092672
531	281961	149721291	23·0434372	8·097758
532	283024	150568768	23·0651252	8·102838
533	284089	151419437	23·0867928	8·107912
534	285156	152273304	23·1084400	8·112980
535	286225	153130375	23·1300670	8·118041
536	287296	153990656	23·1516738	8·123096
537	288369	154854153	23·1732605	8·128144
538	289444	155720872	23·1948270	8·133186
539	290521	156590819	23·2163735	8·138223
540	291600	157464000	23·2379001	8·143253
541	292681	158340421	23·2594067	8·148276
542	293764	159220088	23·2808935	8·153293
543	294849	160103007	23·3023604	8·158304
544	295936	160989184	23·3238076	8·163309
545	297025	161878625	23·3452351	8·168308
546	298116	162771336	23·3666429	8·173302
547	299209	163667323	23·3880311	8·178289
548	300304	164566592	23·4093998	8·183269
549	301401	165469149	23·4307490	8·188244
550	302500	166375000	23·4520788	8·193212

Numb.	Square.	Cube.	Square Root.	Cube Root.
551	303601	167284151	23·4733892	8·198175
552	304704	168196608	23·4946802	8·203131
553	305809	169112377	23·5159520	8·208082
554	306916	170031464	23·5372046	8·213027
555	308025	170953875	23·5584380	8·217965
556	309136	171879616	23·5796522	8·222898
557	310249	172808693	23·6008474	8·227825
558	311364	173741112	23·6220236	8·232746
559	312481	174676879	23·6431808	8·237661
560	313600	175616000	23·6643191	8·242570
561	314721	176558481	23·6854386	8·247474
562	315844	177504328	23·7065392	8·252371
563	316969	178453547	23·7276210	8·257263
564	318096	179406144	23·7486842	8·262149
565	319225	180362125	23·7697286	8·267029
566	320356	181321496	23·7907545	8·271903
567	321489	182284263	23·8117618	8·276772
568	322624	183250432	23·8327506	8·281635
569	323761	184220009	23·8537209	8·286493
570	324900	185193000	23·8746728	8·291344
571	326041	186169411	23·8956063	8·296190
572	327184	187149248	23·9165215	8·301030
573	328329	188132517	23·9374184	8·305865
574	329476	189119224	23·9582971	8·310694
575	330625	190109375	23·9791576	8·315517
576	331776	191102976	24·0000000	8·320335
577	332929	192100033	24·0208243	8·325147
578	334084	193100552	24·0416306	8·329954
579	335241	194104539	24·0624188	8·334755
580	336400	195112000	24·0831892	8·339551
581	337561	196122941	24·1039416	8·344341
582	338724	197137368	24·1246762	8·349125
583	339889	198155287	24·1453929	8·353904
584	341056	199176704	24·1660919	8·358678
585	342225	200201625	24·1867732	8·363446
586	343396	201230056	24·2074369	8·368209
587	344569	202262003	24·2280829	8·372966
588	345744	203297472	24·2487113	8·377718
589	346921	204336469	24·2693222	8·382465
590	348100	205379000	24·2899156	8·387206
591	349281	206425071	24·3104916	8·391942
592	350464	207474688	24·3310501	8·396673
593	351649	208527857	24·3515913	8·401398
594	352836	209584584	24·3721152	8·406118
595	354025	210644875	24·3926218	8·410832
596	355216	211708736	24·4131112	8·415541
597	356409	212776173	24·4335834	8·420245
598	357604	213847192	24·4540385	8·424944
599	358801	214921799	24·4744765	8·429638
600	360000	216000000	24·4948974	8·434327

Numb.	Square.	Cube.	Square Root.	Cube Root.
601	361201	217081801	24·5153013	8·439009
602	362404	218167208	24·5356883	8·443687
603	363609	219256227	24·5560583	8·448360
604	364816	220348864	24·5764115	8·453027
605	366025	221445125	24·5967478	8·457689
606	367236	222545016	24·6170673	8·462347
607	368449	223648543	24.6373700	8·466999
608	369664	224755712	24·6576560	8·471647
609	370881	225866529	24·6779254	8·476289
610	372100	226981000	24·6981781	8·480926
611	373321	228099131	24·7184142	8·485557
612	374544	229220928	24·7386338	8·490184
613	375769	230346397	24·7588368	8·494806
614	376996	231475544	24·7790234	8·499423
615	378225	232608375	24·7991935	8·504034
616	379456	233744896	24·8193473	8·508641
617	380689	234885113	24·8394847	8·513243
618	381924	236029032	24·8596058	8·517840
619	383161	237176659	24·8797106	8·522432
620	384400	238328000	24·8997992	8·527018
621	385641	239483061	24·9198716	8·531600
622	386884	240641848	24·9399278	8·536177
623	388129	241804367	24·9599679	8·540749
624	389376	242970624	24·9799920	8·545317
625	390625	244140625	25·0000000	8·549879
626	391876	245314376	25·0199920	8·554437
627	393129	246491883	25·0399681	8·558990
628	394384	247673152	25·0599282	8·563537
629	395641	248858189	25·0798724	8·568080
630	396900	250047000	25·0998008	8·572618
631	398161	251239591	25·1197134	8·577152
632	399424	252435968	25·1396102	8·581680
633	400689	253636137	25·1594913	8·586204
634	401956	254840104	25·1793566	8·590723
635	403225	256047875	25·1992063	8·595238
636	404496	257259456	25·2190404	8·599747
637	405769	258474853	25·2388589	8·604252
638	407044	259694072	25·2586619	8 608752
639	408321	260917119	25·2784493	8·613248
640	409600	262144000	25·2982213	8·617738
641	410881	263374721	25·3179778	8·622224
642	412164	264609288	25·3377189	8·626706
643	413449	265847707	25·3574447	8·631183
644	414736	267089984	25·3771551	8·635655
645	416025	268336125	25·3968502	8·640122
646	417316	269586136	25·4165301	8·644585
647	418609	270840023	25·4361947	8·649043
648	419904	272097792	25·4558441	8·653497
649	421201	273359449	25·4754784	8·657946
650	422500	274625000	25·4950076	8·662301

Numb.	Square.	Cube.	Square Root.	Cube Root.
651	423801	275894451	25·5147016	8·666831
652	425104	277167808	25·5342907	8·671266
653	426409	278445077	25·5538647	8·675697
654	427716	279726264	25·5734237	8·680123
655	429025	281011375	25·5929678	8·684545
656	430336	282300416	25·6124969	8·688963
657	431649	283593393	25·6320112	8·693376
658	432964	284890312	25·6515107	8·697784
659	434281	285191179	25·6709953	8·702188
660	435600	287496000	25·6904652	8·706587
661	436921	288804781	25·7099203	8·710982
662	438244	290117528	25·7203607	8·715373
663	439569	291434247	25·7487864	8·719759
664	440896	292754944	25·7681975	8·724141
665	442225	294079625	25·7875939	8·728518
666	443556	295408296	25·8069758	8·732891
667	444889	296740963	25·8263431	8·737260
668	446224	298077632	25·8456960	8·741624
669	447561	299418309	25·8650343	8·745984
670	448900	300763000	25·8843582	8·750340
671	450241	302111711	25·9036677	8·754691
672	451584	303464448	25·9229628	8·759038
673	452929	304821217	25·9422435	8·763380
674	454276	306182024	25·9615100	8·767719
675	455625	307546875	25·9807621	8·772053
676	456976	308915776	26·0000000	8·776382
677	458329	310288733	26·0192237	8·780708
678	459684	311665752	26·0384331	8·785029
679	461041	313046839	26·0576284	8·789346
680	462400	314432000	26·0768096	8·793659
681	463761	315821241	26·0959767	8·797967
682	465124	317214568	26·1151297	8·802272
683	466489	318611987	26·1342687	8·806572
684	467856	320013504	26·1533937	8·810868
685	469225	321419125	26·1725047	8·815159
686	470596	322828856	26·1916017	8·819447
687	471969	324242703	26·2106848	8·823730
688	473344	325660672	26·2297541	8·828009
689	474721	327082769	26·2488095	8·832285
690	476100	328509000	26·2678511	8·836556
691	477481	329939371	26·2868789	8·840822
692	478864	331373888	26·3058929	8·845085
693	480249	332812557	26·3248932	8·849344
694	481636	334255384	26·3438797	8·853598
695	483025	335702375	26·3628527	8·857849
696	484416	337153536	26·3818119	8·862095
697	485809	338608873	26·4007576	8·866337
698	487204	340068392	26·4196896	8·870575
699	488601	341532099	26·4386081	8·874809
700	490000	343000000	26·4575131	8·879040

Numb.	Square.	Cube.	Square Root.	Cube Root.
701	491401	344472101	26·4764046	8·883266
702	492804	345948008	26·4952826	8·887488
703	494209	347428927	26·5141472	8·891706
704	495616	348913664	26·5329983	8·895920
705	497025	350402625	26·5518361	8·900130
706	498436	351895816	26·5706605	8·904336
707	499849	353393243	26·5894716	8·908538
708	501264	354894912	26·6082694	8·912736
709	502681	356400829	26·6270539	8·916931
710	504100	357911000	26·6458252	8·921121
711	505521	359425431	26·6645833	8·925307
712	506944	360944128	26·6833281	8·929490
713	508369	362467097	26·7020598	8·933668
714	509796	363994344	26·7207784	8·937843
715	511225	365525875	26·7394839	8·942014
716	512656	367061696	26·7581763	8·946180
717	514089	368601813	26·7768557	8·950343
718	515524	370146232	26·7955220	8·954502
719	516961	371694959	26·8141754	8·958658
720	518400	373248000	26·8328157	8·962809
721	519841	374805361	26·8514432	8·966957
722	521284	376367048	26·8700577	8·971100
723	522729	377933067	26·8886593	8·975240
724	524176	379503424	26·9072481	8·979376
725	525625	381078125	26·9258240	8·983508
726	527076	382657176	26·9443872	8·987637
727	528529	384240583	26·9629375	8·991762
728	529984	385828352	26·9814751	8·995883
729	531441	387420489	27·0000000	9·000000
730	532900	389017000	27·0185122	9·004113
731	534361	390617891	27·0370117	9·008222
732	535824	392223168	27·0554985	9·012328
733	537289	393832837	27·0739727	9·016430
734	538756	395446904	27·0924344	9·020529
735	540225	397065375	27·1108834	9·024623
736	541696	398688256	27·1293199	9·028714
737	543169	400315553	27·1477439	9·032802
738	544644	401947272	27·1661554	9·036885
739	546121	403583419	27·1845544	9·040965
740	547600	405224000	27·2029410	9·045041
741	549081	406869021	27·2213152	9·049114
742	550564	408518488	27·2396769	9·053183
743	552049	410172407	27·2580263	9·057248
744	553536	411830784	27·2763634	9·061309
745	555025	413493625	27·2946881	9·065367
746	556516	415160936	27·3130006	9·069422
747	558009	416832723	27·3313007	9·073472
748	559504	418508992	27·3495887	9·077519
749	561001	420189749	27·3678644	9·081563
750	562500	421875000	27·3861279	9·085603

Numb.	Square.	Cube.	Square Root.	Cube Root.
751	564001	423564751	27·4043792	9·089639
752	565504	425259008	27·4226184	9·093672
753	567009	426957777	27·4408455	9·097701
754	568516	428661064	27·4590604	9·101726
755	570025	430368875	27·4772633	9·105748
756	571536	432081216	27·4954542	9·109766
757	573049	433798093	27·5136330	9·113781
758	574564	435519512	27·5317998	9·117793
759	576081	437245479	27·5499546	9·121801
760	577600	438976000	27·5680975	9.125805
761	579121	440711081	27·5862284	9·129806
762	580644	442450728	27·6043475	9·133803
763	582169	444194947	27·6224546	9·137797
764	583696	445943744	27·6405499	9·141788
765	585225	447697125	27·6586334	9·145774
766	586756	449455096	27·6767050	9·149757
767	588289	451217663	27·6947648	9·153737
768	589824	452984832	27·7128129	9·157713
769	591361	454756609	27·7308492	9·161686
770	592900	456533000	27·7488739	9·165656
771	594441	458314011	27·7668868	9·169622
772	595984	460099648	27·7848880	9·173585
773	597529	461889917	27·8028775	9·177544
774	599076	463684824	27·8208555	9·181500
775	600625	465484375	27·8388218	9·185452
776	602176	467288576	27·8567766	9·189401
777	603729	469097433	27·8747197	9·193347
778	605284	470910952	27·8926514	9·197289
779	606841	472729139	27·9105715	9·201228
780	608400	474552000	27·9284801	9·205164
781	609961	476379541	27·9463772	9·209096
782	611524	478211768	27·9642629	9·213025
783	613089	480048687	27·9821372	9·216950
784	614656	481890304	28·0000000	9·220872
785	616225	483736025	28·0178515	9·224791
786	617796	485587656	28·0356915	9·228706
787	619369	487443403	28·0535203	9·232618
788	620944	489303872	28·0713377	9·237527
789	622521	491169069	28·0891438	9·240433
790	624100	493039000	28·1069386	9·244335
791	625681	494913671	28·1247222	9·248234
792	627264	496793088	28·1424946	9·252130
793	628849	498677257	28·1602557	9·256022
794	630436	500566184	28·1780056	9·259911
795	632025	502459875	28·1957444	9·263797
796	633616	504358336	28·2134720	9·267679
797	635209	506261573	28·2311884	9·271559
798	636804	508169592	28·2488938	9·275435
799	638401	510082399	28·2665881	9·279308
800	640000	512000000	28·2842712	9·283177

Numb	Square.	Cube.	Square Root.	Cube Root.
801	641601	513922401	28·3019434	9·287044
802	643204	515849608	28·3196045	9·290907
803	644809	517781627	28·3372546	9·294767
804	646416	519718464	28·3548938	9·298623
805	648025	521660125	28·3725219	9·302477
806	649636	523606616	28·3901391	9·306327
807	651249	525557943	28·4077454	9·310175
808	652864	527514112	28·4253408	9·314019
809	654481	529475129	28·4429253	9·317859
810	656100	531441000	28·4604989	9·321697
811	657721	533411731	28·4780617	9·325532
812	659344	535387328	28·4956137	9·329363
813	660969	537366797	28·5131549	9·333191
814	662596	539353144	28·5306852	9·337016
815	664225	541343375	28·5482048	9·340838
816	665856	543338496	28·5657137	9·344657
817	667489	545338513	28·5832119	9·348473
818	669124	547343432	28·6006993	9·352285
819	670761	549353259	28·6181760	9·356095
820	672400	551368000	28·6356421	9·359901
821	674041	553387661	28·6530976	9·363704
822	675684	555412248	28·6705424	9·367505
823	677329	557441767	28·6879766	9·371302
824	678976	559476224	28·7054002	9·375096
825	680625	561515625	28·7228132	9·378887
826	682276	563559976	28·7402157	9·382675
827	683929	565609283	28·7576077	9·386460
828	685584	567663552	28·7749891	9·390241
829	687241	569722789	28·7923601	9·394020
830	688900	571787000	28·8097206	9·397796
831	690561	573856191	28·8270706	9·401569
832	692224	575930368	28·8444102	9·405338
833	693889	578009537	28·8617394	9·409105
834	695556	580093704	28·8790582	9·412869
835	697225	582182875	28·8963666	9·416630
836	698896	584277056	28·9136646	9·420387
837	700569	586376253	28·9309523	9·424141
838	702244	588480472	28·9482297	9·427893
839	703921	590589719	28·9654967	9·431642
840	705600	592704000	28·9827535	9·435388
841	707281	594823321	29·0000000	9·439130
842	708964	596947688	29·0172363	9·442870
843	710649	599077107	29·0344623	9·446607
844	712336	601211584	29·0516781	9·450341
845	714025	603351125	29·0688837	9·454071
846	715716	605495736	29·0860791	9·457799
847	717409	607645423	29·1032644	9·461524
848	719104	609800192	29·1204396	9·465247
849	720801	611960049	29·1376046	9·468966
850	722500	614125000	29·1547595	9·472682

Numb.	Square.	Cube.	Square Root.	Cube Root.
851	724201	616295051	29·1719043	9·476395
852	725904	618470208	29·1890390	9·480106
853	727609	620650477	29·2061637	9·483813
854	729316	622835864	29·2232784	9·487518
855	731025	625026375	29·2403830	9·491219
856	732736	627222016	29·2574777	9·494918
857	734449	629422793	29·2745623	9·498614
858	736164	631628712	29·2916370	9·502307
859	737881	633839779	29·3087018	9·505998
860	739600	636056000	29·3257566	9·509685
861	741321	638277381	29·3428015	9·513369
862	743044	640503928	29·3598365	9·517051
863	744769	642735647	29·3768616	9·520730
864	746496	644972544	29·3938769	9·524406
865	748225	647214625	29·4108823	9·528079
866	749956	649461896	29·4278779	9·531749
867	751689	651714363	29·4448637	9·535417
868	753424	653972032	29·4618397	9·539081
869	755161	656234909	29·4788059	9·542743
870	756900	658503000	29·4957624	9·546402
871	758641	660776311	29·5127091	9·550058
872	760384	663054848	29·5296461	9·553712
873	762129	665338617	29·5465734	9·557363
874	763876	667627624	29·5634910	9·561010
875	765625	669921875	29·5803989	9·564655
876	767376	672221376	29·5972972	9·568297
877	769129	674526133	29·6141858	9·571937
878	770884	676836152	29·6310648	9·575574
879	772641	679151439	29·6479325	9·579208
880	774400	681472000	29·6647989	9·582839
881	776161	683797841	29·6816442	9·586468
882	777924	686128968	29·6984848	9·590093
883	779689	688465387	29·7153159	9·593716
884	781456	690807104	29·7321375	9·597337
385	783225	693154125	29·7489496	9·600954
886	784996	695506456	29·7657521	9·604569
887	786769	697864103	29·7825452	9·608181
888	788544	700227072	29·7993289	9·611791
889	790321	702595369	29·8161030	9·615397
890	792100	704969000	29·8328678	9·619001
891	793881	707347971	29·8496231	9·622603
892	795664	709732288	29·8663690	9·626201
893	797449	712121957	29·8831056	9·629797
894	799236	714516984	29·8998328	9·633390
895	801025	716917375	29·9165506	9·636981
896	802816	719323136	29·9332591	9·640569
897	804609	721734273	29·9499583	9·644154
898	806404	724150792	29·9666481	9·647736
899	808201	726572699	29·9833287	9·651316
900	810000	729000000	30·0000000	9·654893

Numb.	Square.	Cube.	Square Root.	Cube Root
901	811801	731432701	30·0166620	9·658468
902	813604	733870808	30·0333148	9·662040
903	815409	736314327	30·0499584	9·665609
904	817216	738763264	30·0665928	9·669176
905	819025	741217625	30·0832179	9·672740
906	820836	743677416	30·0998339	9·676301
907	822649	746142643	30·1164407	9·679860
908	824464	748613312	30·1330383	9·683416
909	826281	751089429	30·1496269	9·686970
910	828100	753571000	30·1632063	9·690521
911	829921	756058031	30·1827765	9·694069
912	831744	758550528	30·1993377	9·697615
913	833569	761048497	30·2158899	9·701158
914	835396	763551944	30·2324329	9·704698
915	837225	766060875	30·2489669	9·708236
916	839056	768575296	30·2654919	9·711772
917	840889	771095213	30·2820079	9·715305
918	842724	773620632	30·2985148	9·718835
919	844561	776151559	30·3150128	9·722363
920	846400	778688000	30·3315018	9·725888
921	848241	781229961	30·3479818	9·729410
922	850084	783777448	30·3644529	9·732930
923	851929	786330467	30·3809151	9·736448
924	853776	788889024	30·3973683	9·739953
925	855625	791453125	30·4138127	9·743475
926	857476	794022776	30·4302481	9·746985
927	859329	796597983	30·4466747	9·750493
928	861184	799178752	30·4630924	9·753998
929	863041	801765089	30·4795013	9·757500
930	864900	804357000	30·4959014	9·761000
931	866761	806954491	30·5122926	9·764497
932	868624	809557568	30·5286750	9·767992
933	870489	812166237	30·5450487	9·771484
934	872356	814780504	30·5614136	9·774974
935	874225	817400375	30·5777697	9·778461
936	876096	820025856	30·5941171	9·781946
937	877969	822656953	30·6104557	9·785423
938	879844	825293672	30·6267857	9·788908
939	881721	827936019	30·6431069	9·792386
940	883600	830584000	30·6594194	9·795861
941	885481	833237621	30·6757233	9·799333
942	887364	835896888	30·6920185	9·802803
943	889249	838561807	30·7083051	9·806271
944	891136	841232384	30·7245830	9·809736
945	893025	843908625	30·7408523	9·813198
946	894916	846590536	30·7571130	9·816659
947	896809	849278123	30·7733651	9·820117
948	898704	851971392	30·7896086	9·823572
949	900601	854670349	30·8058436	9·827025
950	902500	857375000	30·8220700	9·830475

Numb.	Square.	Cube.	Square Root.	Cube Root.
851	724201	616295051	29·1719043	9·476395
852	725904	618470208	29·1890390	9·480106
853	727609	620650477	29·2061637	9·483813
854	729316	622835864	29·2232784	9·487518
855	731025	625026375	29·2403830	9·491219
856	732736	627222016	29·2574777	9·494918
857	734449	629422793	29·2745623	9·498614
858	736164	631628712	29·2916370	9·502307
859	737881	633839779	29·3087018	9·505998
860	739600	636056000	29·3257566	9·509685
861	741321	638277381	29·3428015	9·513369
862	743044	640503928	29·3598365	9·517051
863	744769	642735647	29·3768616	9·520730
864	746496	644972544	29·3938769	9·524406
865	748225	647214625	29·4108823	9·528079
866	749956	649461896	29·4278779	9·531749
867	751689	651714363	29·4448637	9·535417
868	753424	653972032	29·4618397	9·539081
869	755161	656234909	29·4788059	9·542743
870	756900	658503000	29·4957624	9·546402
871	758641	660776311	29·5127091	9·550058
872	760384	663054848	29·5296461	9·553712
873	762129	665338617	29·5465734	9·557363
874	763876	667627624	29·5634910	9·561010
875	765625	669921875	29·5803989	9·564655
876	767376	672221376	29·5972972	9·568297
877	769129	674526133	29·6141858	9·571937
878	770884	676836152	29·6310648	9·575574
879	772641	679151439	29·6479325	9·579208
880	774400	681472000	29·6647939	9·582839
881	776161	683797841	29·6816442	9·586468
882	777924	686128968	29·6984848	9·590093
883	779689	688465387	29·7153159	9·593716
884	781456	690807104	29·7321375	9·597337
385	783225	693154125	29·7489496	9·600954
886	784996	695506456	29·7657521	9·604569
887	786769	697864103	29·7825452	9·608181
888	788544	700227072	29·7993289	9·611791
889	790321	702595369	29·8161030	9·615397
890	792100	704969000	29·8328678	9·619001
891	793881	707347971	29·8496231	9·622603
892	795664	709732288	29·8663690	9·626201
893	797449	712121957	29·8831056	9·629797
894	799236	714516984	29·8998328	9·633390
895	801025	716917375	29·9165506	9·636981
896	802816	719323136	29·9332591	9·640569
897	804609	721734273	29·9499583	9·644151
898	806404	724150792	29·9666481	9·647736
899	808201	726572699	29·9833287	9·651316
900	810000	729000000	30·0000000	9·654893

Numb.	Square.	Cube.	Square Root.	Cube Root.
901	811801	731432701	30·0166620	9·658468
902	813604	733870808	30·0333148	9·662040
903	815409	736314327	30·0499584	9·665609
904	817216	738763264	30·0665928	9·669176
905	819025	741217625	30·0832179	9·672740
906	820836	743677416	30·0998339	9·676301
907	822649	746142643	30·1164407	9·679860
908	824464	748613312	30·1330383	9·683416
909	826281	751089429	30·1496269	9·686970
910	828100	753571000	30·1662063	9·690521
911	829921	756058031	30·1827765	9·694069
912	831744	758550528	30·1993377	9·697615
913	833569	761048497	30·2158899	9·701158
914	835396	763551944	30·2324329	9·704698
915	837225	766060875	30·2489669	9·708236
916	839056	768575296	30·2654919	9·711772
917	840889	771095213	30·2820079	9·715305
918	842724	773620632	30·2985148	9·718835
919	844561	776151559	30·3150128	9·722363
920	846400	778688000	30·3315018	9·725888
921	848241	781229961	30·3479818	9·729410
922	850084	783777448	30·3644529	9·732930
923	851929	786330467	30·3809151	9·736448
924	853776	788889024	30·3973683	9·739963
925	855625	791453125	30·4138127	9·743475
926	857476	794022776	30·4302481	9·746985
927	859329	796597983	30·4466747	9·750493
928	861184	799178752	30·4630924	9·753998
929	863041	801765089	30·4795013	9·757500
930	864900	804357000	30·4959014	9·761000
931	866761	806954491	30·5122926	9·764497
932	868624	809557568	30·5286750	9·767992
933	870489	812166237	30·5450487	9·771484
934	872356	814780504	30·5614136	9·774974
935	874225	817400375	30·5777697	9·778461
936	876096	820025856	30·5941171	9·782946
937	877969	822656953	30·6104557	9·785423
938	879844	825293672	30·6267857	9·788908
939	881721	827936019	30·6431069	9·792386
940	883600	830584000	30·6594194	9·795861
941	885481	833237621	30·6757233	9·799333
942	887364	835896888	30·6920185	9·802803
943	889249	838561807	30·7083051	9·806271
944	891136	841232384	30·7245830	9·809736
945	893025	843908625	30·7408523	9·813198
946	894916	846590536	30·7571130	9·816659
947	896809	849278123	30·7733651	9·820117
948	898704	851971392	30·7896086	9·823572
949	900601	854670349	30·8058436	9·827025
950	902500	857375000	30·8220700	9·830475

Numb.	Square.	Cube.	Square Root.	Cube Root
951	904401	860085351	30·8382879	9·833923
952	906304	862801408	30·8544972	9·837369
953	908209	865523177	30·8706981	9·840812
954	910116	868250664	30·8868904	9·844253
955	912025	870983875	30·9030743	9·847692
956	913936	873722816	30·9192497	9·851128
957	915849	876467493	30·9354166	9·854561
958	917764	879217912	30 9515751	9·857992
959	919681	881974079	30·9677251	9·861421
960	921600	884736000	30·9838668	9·861848
961	923521	887503681	31·0000000	9·868272
962	925444	890277128	31·0161248	9·871694
963	927369	893056347	31·0322413	9·875113
964	929296	895841344	31·0483494	9·878530
965	931225	898632125	31·0644491	9·881945
966	933156	901428696	31·0805405	9·885357
967	935089	904231063	31·0966236	9·888767
968	937024	907039232	31·1126984	9·892174
969	938961	909853209	31·1287648	9·895580
970	940900	912673000	31·1448230	9·898983
971	942841	915498611	31·1608729	9·902383
972	944784	918330048	31·1769145	9·905781
973	946729	921167317	31·1929479	9·909177
974	948676	924010424	31·2089731	9·912571
975	950625	926859375	31·2249900	9·915962
976	952576	929714176	31·2409987	9·919351
977	954529	932574833	31·2569992	9·922738
978	956484	935441352	31·2729915	9·926122
979	958441	938313739	31·2889757	9·929504
980	960400	941192001	31·3049517	9·932883
981	962361	944076141	31·3209195	9·936261
982	964324	946966168	31·3368792	9·939636
983	966289	949862087	31·3528308	9·943009
984	968256	952763904	31·3687743	9·946379
985	970225	955671625	31·3847097	9·949747
986	972196	958585256	31·4006369	9·953113
987	974169	961504803	31·4165561	9·956477
988	976144	964430272	31·4324673	9·959839
989	978121	967361669	31·4483704	9·963198
990	980100	970299000	31·4642654	9·966554
991	982081	973242271	31·4801525	9·969909
992	984064	976191488	31·4960315	9·973262
993	986049	979146657	31·5119025	9·976612
994	988036	982107784	31·5277655	9·979959
995	990025	985074875	31·5436206	9·983304
996	992016	988047936	31·5594677	9·986648
997	994009	991026973	31·5753068	9·989990
998	996004	994011992	31·5911380	9·993328
999	998001	997002999	31·6069613	9·996665

OF RATIOS, PROPORTIONS, AND PROGRESSIONS.

NUMBERS are compared to each other in two different ways: the one comparison considers the difference of the two numbers, and is named Arithmetical Relation; and the difference sometimes the Arithmetical Ratio: the other considers their quotient, which is called Geometrical Relation; and the quotient is the Geometrical Ratio. So, of these two numbers 6 and 3, the difference, or arithmetical ratio, is $6 - 3$ or 3, but the geometrical ratio is $\frac{6}{3}$ or 2.

There must be two numbers to form a comparison: the number which is compared, being placed first, is called the Antecedent; and that to which it is compared, the Consequent. So, in the two numbers above, 6 is the antecedent, and 3 the consequent.

If two or more couplets of numbers have equal ratios, or equal differences, the equality is named Proportion, and the terms of the ratios Proportionals. So, the two couplets, 4, 2 and 8, 6, are arithmetical proportionals, because $4 - 2 = 8 - 6 = 2$; and the two couplets 4, 2 and 6, 3, are geometrical proportionals, because $\frac{4}{2} = \frac{6}{3} = 2$, the same ratio.

To denote numbers as being geometrically proportional, a colon is set between the terms of each couplet, to denote their ratio; and a double colon, or else a mark of equality, between the couplets or ratios. So, the four proportionals, 4, 2, 6, 3 are set thus, $4 : 2 :: 6 : 3$, which means, that 4 is to 2 as 6 is to 3; or thus, $4 : 2 = 6 : 3$, or thus, $\frac{4}{2} = \frac{6}{3}$, both which mean, that the ratio of 4 to 2, is equal to the ratio of 6 to 3.

Proportion is distinguished into Continued and Discontinued. When the difference or ratio of the consequent of one couplet, and the antecedent of the next couplet, is not the same as the common difference or ratio of the couplets, the proportion is discontinued. So, 4, 2, 8, 6 are in discontinued arithmetical proportion, because $4 - 2 = 8 - 6 = 2$, whereas $8 - 2 = 6$: and 4, 2, 6, 3 are in discontinued geometrical proportion, because $\frac{4}{2} = \frac{6}{3} = 2$, but $\frac{6}{2} = 3$, which is not the same.

But when the difference or ratio of every two succeeding terms is the same quantity, the proportion is said to be Continued, and the numbers themselves make a series of Continued Proportionals,

Proportionals, or a progression. So 2, 4, 6, 8 form an arithmetical progression, because $4-2=6-4=8-6=2$, all the same common difference; and 2, 4, 8, 16 a geometrical progression, because $\frac{4}{2}=\frac{8}{4}=\frac{16}{8}=2$, all the same ratio.

When the following terms of a progression increase, or exceed each other, it is called an Ascending Progression, or Series; but when the terms decrease, it is a descending one. So, 0, 1, 2, 3, 4, &c. is an ascending arithmetical progression, but 9, 7, 5, 3, 1, &c. is a descending arithmetical progression. Also 1, 2, 4, 8, 16, &c. is an ascending geometrical progression, and 16, 8, 4, 2, 1, &c. is a descending geometrical progression.

ARITHMETICAL PROPORTION *and* PROGRESSION.

In Arithmetical Progression, the numbers or terms have all the same common difference. Also, the first and last terms of a Progression, are called the Extremes; and the other terms, lying between them, the Means. The mose useful part of arithmetical proportions, is contained in the following theorems:

Theorem 1. When four quantities are in arithmetical proportion, the sum of the two extremes is equal to the sum of the two means. Thus, of the four 2, 4, 6, 8, here $2+8=4+6=10$.

Theorem 2. In any continued arithmetical progression, the sum of the two extremes is equal to the sum of any two means that are equally distant from them, or equal to double the middle term when there is an uneven number of terms.

Thus, in the terms 1, 3, 5, it is $1+5=3+3=6$.

And in the series 2, 4, 6, 8, 10, 12, 14, it is $2+14=4+12=6+10=8+8=16$.

Theorem 3. The difference between the extreme terms of an arithmetical progression, is equal to the common difference of the series multiplied by one less than the number of the terms. So, of the ten terms, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, the common difference is 2, and one less than the number of terms 9; then the difference of the extremes is $20-2=18$, and $2\times 9=18$ also.

Consequently,

Consequently the greatest term is equal to the least term added to the product of the common difference multiplied by 1 less than the number of terms.

THEOREM 4. The sum of all the terms, of any arithmetical progression, is equal to the sum of the two extremes multiplied by the number of terms, and divided by 2; or the sum of the two extremes multiplied by the number of the terms, gives double the sum of all the terms in the series.

This is made evident by setting the terms of the series in an inverted order, under the same series in a direct order, and adding the corresponding terms together in that order. Thus,

in the series	1,	3,	5,	7,	9,	11,	13,	15;
ditto inverted	15,	13,	11,	9,	7,	5,	3,	1;
the sums are	16 +	16 +	16 +	16 +	16 +	16 +	16 +	16,

which must be double the sum of the single series, and is equal to the sum of the extremes repeated as often as are the number of the terms.

From these theorems may readily be found any one of these five parts; the two extremes, the number of terms, the common difference, and the sum of all the terms, when any three of them are given; as in the following problems:

PROBLEM I.

Given the Extremes, and the Number of Terms; to find the Sum of all the Terms.

ADD the extremes together, multiply the sum by the number of terms, and divide by 2.

EXAMPLES.

1. The extremes being 3 and 19, and the number of terms 9; required the sum of the terms?

$$\begin{array}{r} 19 \\ 3 \\ \hline 22 \\ 9 \\ \hline 2\,)\;198 \\ \hline \text{Ans. } 99 \end{array}$$

Or, $\frac{19+3}{2} \times 9 = \frac{22}{2} \times 9 = 11 \times 9 = 99$, the same answer.

2. It is required to find the number of all the strokes a common clock strikes in one whole revolution of the index, or in 12 hours? Ans. 78.

Ex.

Ex. 3. How many strokes do the clocks of Venice strike in the compass of the day, which go continually on from 1 to 24 o'clock? Ans. 300.

4. What debt can be discharged in a year, by weekly payments in arithmetical progression, the first payment being 1*s*, and the last or 52d payment 5*l* 3*s*? Ans. 135*l* 4*s*.

PROBLEM II.

Given the Extremes, and the Number of Terms; to find the Common Difference.

SUBTRACT the less extreme from the greater, and divide the remainder by 1 less than the number of terms, for the common difference.

EXAMPLES.

1. The extremes being 3 and 19, and the number of terms 9; required the common difference?

$$\begin{array}{r} 19 \\ 3 \\ \hline 8)\ 16 \\ \hline \text{Ans.}\ \ 2 \\ \hline \end{array} \qquad \text{Or, } \frac{19-3}{9-1} = \frac{16}{8} = 2.$$

2. If the extremes be 10 and 70, and the number of terms 21; what is the common difference, and the sum of the series? Ans. the com. diff. is 3, and the sum is 840.

3. A certain debt can be discharged in one year, by weekly payments in arithmetical progression, the first payment being 1*s*, and the last 5*l* 3*s*; what is the common difference of the terms? Ans. 2.

PROBLEM III.

Given one of the Extremes, the Common Difference, and the Number of Terms: to find the other Extreme, and the Sum of the Series.

MULTIPLY the common difference by 1 less than the number of terms, and the product will be the difference of the extremes: Therefore add the product to the less extreme, to give the greater; or subtract it from the greater, to give the less extreme.

EXAMPLES.

1. Given the least term 3, the common difference 2, of an arithmetical series of 9 terms; to find the greatest term, and the sum of the series.

2	
8	
16	
3	
19	the greatest term
3	the least
22	sum
9	number of terms.
2) 198	
99	the sum of the series.

2. If the greatest term be 70, the common difference 3, and the number of terms 21, what is the least term, and the sum of the series?

Ans. The least term is 10, and the sum is 840.

3. A debt can be discharged in a year, by paying 1 shilling the first week, 3 shillings the second, and so on, always 2 shillings more every week; what is the debt, and what will the last payment be?

Ans. The last payment will be *5l* *3s*, and the debt is 135*l* 4*s*.

PROBLEM IV.

To find an Arithmetical Mean Proportional between Two Given Terms.

Add the two given extremes or terms together, and take half their sum for the arithmetical mean required.

EXAMPLE.

To find an arithmetical mean between the two numbers 4 and 14.

Here	
14	
4	
2) 18	
Ans. 9	the mean required.

PROBLEM

PROBLEM V.

To find Two Arithmetical Means between Two Given Extremes.

SUBTRACT the less extreme from the greater, and divide the difference by 3, so will the quotient be the common difference; which being continually added to the less extreme, or taken from the greater, gives the means.

EXAMPLE.

To find two arithmetical means between 2 and 8.

Here 8
2
―――
3) 6
―――
com. dif. 2

Then $2 + 2 = 4$ the one mean.
and $4 + 2 = 6$ the other mean.

PROBLEM VI.

To find any Number of Arithmetical Means between Two Given Terms or Extremes.

SUBTRACT the less extreme from the greater, and divide the difference by 1 more than the number of means required to be found, which will give the common difference; then this being added continually to the least term, or subtracted from the greatest, will give the mean terms required.

EXAMPLE.

To find five arithmetical means between 2 and 14.

Here 14
2
―――
6) 12
―――
com. dif. 2

Then by adding this com. dif. continually, the means are found 4, 6, 8, 10, 12.

See more of Arithmetical progression in the Algebra.

GEOMETRICAL PROPORTION *and* PROGRESSION.

In Geometrical Progression the numbers or terms have all the same multiplier or divisor. The most useful part of Geometrical Proportion, is contained in the following theorems.

Theorem 1. When four quantities are in geometrical proportion, the product of the two extremes is equal to the product of the two means.

Thus, in the four 2, 4, 3, 6, it is $2 \times 6 = 3 \times 4 = 12$.

And hence, if the product of the two means be divided by one of the extremes, the quotient will give the other extreme. So, of the above numbers, the product of the means $12 \div 2 = 6$ the one extreme, and $12 \div 6 = 2$ the other extreme; and this is the foundation and reason of the practice in the Rule of Three.

Theorem 2. In any continued geometrical progression, the product of the two extremes is equal to the product of any two means that are equally distant from them, or equal to the square of the middle term when there is an uneven number of terms.

Thus, in the terms 2, 4, 8, it is $2 \times 8 = 4 \times 4 = 16$.

And in the series 2, 4, 8, 16, 32, 64, 128,
it is $2 \times 128 = 4 \times 64 = 8 \times 32 = 16 \times 16 = 256$.

Theorem 3. The quotient of the extreme terms of a geometrical progression, is equal to the common ratio of the series raised to the power denoted by 1 less than the number of the terms. Consequently the greatest term is equal to the least term multiplied by the said quotient.

So, of the ten terms 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, the common ratio is 2, and one less than the number of terms is 9; then the quotient of the extremes is $1024 \div 2 = 512$, and $2^9 = 512$ also.

Theorem

THEOREM 4. The sum of all the terms, of any geometrical progression, is found by adding the greatest term to the difference of the extremes divided by 1 less than the ratio.

So, the sum of 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, (whose ratio is 2), is $1024 + \frac{1024-2}{2-1} = 1024 + 1022 = 2046$.

The foregoing, and several other properties of geometrical proportion, are demonstrated more at large in the Algebraic part of this work. A few examples may here be added of the theorems, just delivered, with some problems concerning mean proportionals.

EXAMPLES.

1. The least of ten terms, in geometrical progression, being 1, and the ratio 2; what is the greatest term, and the sum of all the terms?

Ans. The greatest term is 512, and the sum 1023.

2. What debt may be discharged in a year, or 12 months, by paying 1*l* the first month, 2*l* the second, 4*l* the third, and so on, each succeeding payment being double the last; and what will the last payment be?

Ans. The debt 4095*l*, and the last payment 2048*l*.

PROBLEM I.

To find One Geometrical Mean Proportional between any Two Numbers.

MULTIPLY the two numbers together, and extract the square root of the product, which will give the mean proportional sought.

EXAMPLE.

To find a geometrical mean between the two numbers 3 and 12.

$$\begin{array}{r} 12 \\ 3 \\ \hline 36 \ (\ 6 \text{ the mean.} \\ \hline 36 \end{array}$$

PROBLEM

PROBLEM II.

To find Two Geometrical Mean Proportionals between any Two Numbers.

Divide the greater number by the less, and extract the cube root of the quotient, which will give the common ratio of the terms. Then multiply the least given term by the ratio for the first mean, and this mean again by the ratio for the second mean: or, divide the greater of the two given terms by the ratio for the greater mean, and divide this again by the ratio for the less mean.

EXAMPLE.

To find two geometrical means between 3 and 24.

Here 3) 24 (8; its cube root 2 is the ratio.
Then $3 \times 2 = 6$, and $6 \times 2 = 12$, the two means.
Or $24 \div 2 = 12$, and $12 \div 2 = 6$, the same.
That is, the two means between 3 and 24, are 6 and 12.

PROBLEM III.

To find any Number of Geometrical Means between Two Numbers.

Divide the greater number by the less, and extract such root of the quotient whose index is 1 more than the number of means required; that is, the 2d root for one mean, the 3d root for two means, the 4th root for three means, and so on; and that root will be the common ratio of all the terms. Then, with the ratio, multiply continually from the first term, or divide continually from the last or greatest term.

EXAMPLE.

To find four geometrical means between 3 and 96.

Here 3) 96 (32; the 5th root of which is 2, the ratio.
Then $3 \times 2 = 6$, & $6 \times 2 = 12$, & $12 \times 2 = 24$, & $24 \times 2 = 48$.
Or $96 \div 2 = 48$, & $48 \div 2 = 24$, & $24 \div 2 = 12$, & $12 \div 2 = 6$.
That is, 6, 12, 24, 48, are the four means between 3 and 96.

Of

Of MUSICAL PROPORTION.

There is also a third kind of proportion, called Musical, which being but of little or no common use, a very short account of it may here suffice.

Musical Proportion is when, of three numbers, the first has the same proportion to the third, as the difference between the first and second, has to the difference between the second and third.

As in these three, 6, 8, 12;

where $6 : 12 :: 8 - 6 : 12 - 8$,

that is $6 : 12 :: 2 : 4$.

When four numbers are in musical proportion; then the first has the same ratio to the fourth, as the difference between the first and second has to the difference between the third and fourth.

As in these, 6, 8, 12, 18;

where $6 : 18 :: 8 - 6 : 18 - 12$,

that is $6 : 18 :: 2 : 6$.

When numbers are in musical progression, their reciprocals are in arithmetical progression; and the converse, that is, when numbers are in arithmetical progression, their reciprocals are in musical progression.

So in these musicals 6, 8, 12, their reciprocals $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{12}$, are in arithmetical progression; for $\frac{1}{6} + \frac{1}{12} = \frac{3}{12} = \frac{1}{4}$; and $\frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \frac{1}{4}$; that is, the sum of the extremes is equal to double the mean, which is the property of arithmeticals.

The method of finding out numbers in musical proportion is best expressed by letters in Algebra.

FELLOWSHIP, or PARTNERSHIP.

Fellowship is a rule, by which any sum or quantity may be divided into any number of parts, which shall be in any given proportion to one another.

By this rule are adjusted the gains or loss or charges of

partners

partners in company; or the effects of bankrupts, or legacies in case of a deficiency of assets or effects; or the shares of prizes; or the numbers of men to form certain detachments; or the division of waste lands among a number of proprietors.

Fellowship is either Single or Double. It is Single, when the shares or portions are to be proportional each to one single given number only; as when the stocks of partners are all employed for the same time: And Double, when each portion is to be proportional to two or more numbers; as when the stocks of partners are employed for different times.

SINGLE FELLOWSHIP.

GENERAL RULE.

ADD together the numbers that denote the proportion of the shares. Then say,

As the sum of the said proportional numbers,
Is to the whole sum to be parted or divided,
So is each several proportional number,
To the corresponding share or part.

Or, as the whole stock, is to the whole gain or loss,
So is each man's particular stock,
To his particular share of the gain or loss.

TO PROVE THE WORK. Add all the shares or parts together, and the sum will be equal to the whole number to be shared, when the work is right.

EXAMPLES.

1. To divide the number 240 into three such parts, as shall be in proportion to each other as the three numbers 1, 2 and 3.

Here $1 + 2 + 3 = 6$, the sum of the numbers,

Then, as 6 : 240 :: 1 : 40 the 1st part,
and as 6 : 240 :: 2 : 80 the 2d part,
also as 6 : 240 :: 3 : 120 the 3d part,

Sum of all 240, the proof.

Ex. 2.

Ex. 2. Three persons, A, B, C, freighted a ship with 340 tuns of wine; of which, A loaded 110 tuns, B 97, and C the rest: in a storm the seamen were obliged to throw overboard 85 tuns; how much must each person sustain of the loss?

Here 110 + 97 = 207 tuns, loaded by A and B;
theref. 340 − 207 = 133 tuns, loaded by C.
Hence, as 340 : 85 :: 110
or as 4 : 1 :: 110 : $27\frac{1}{2}$ tuns = A's loss;
and as 4 : 1 :: 97 : $24\frac{1}{4}$ tuns = B's loss;
also as 4 : 1 :: 133 : $33\frac{1}{4}$ tuns = C's loss;

Sum 85 tuns, the proof.

3. Two merchants, C and D, made a stock of 120*l*; of which C contributed 75*l*, and D the rest: by trading they gained 30*l*; what must each have of it?

Ans. C 18*l* 15*s*, and D 11*l* 5*s*.

4. Three merchants, E, F, G, make a stock of 700*l*, of which E contributed 123*l*, F 358*l*, and G the rest: by trading they gain 125*l* 10*s*; what must each have of it?

Ans. E must have	22*l*	1*s*	0*d*	$2\frac{2}{35}q$.
F - - -	64	3	8	$0\frac{32}{35}$.
G - - -	39	5	3	$1\frac{1}{35}$.

5. A General imposing a contribution * of 700*l* on four villages, to be paid in proportion to the number of inhabitants contained in each; the 1st containing 250, the 2d 350, the 3d 400, and the 4th 500 persons; what part must each village pay?

Ans. the 1st to pay	116*l*	13*s*	4*d*.
the 2d - -	163	6	8
the 3d - -	186	13	4
the 4th - -	233	6	8

6. A piece of ground, consisting of 37 ac 2 ro 14 ps, is to be divided among three persons, L, M, and N, in proportion to their estates: now if L's estate be worth 500*l* a year, M's 320*l*, and N's 75*l*; what quantity of land must each one have?

Ans. L must have	20 ac	3 ro	$39\frac{119}{179}$ ps.
M - - -	13	1	$30\frac{46}{179}$.
N - - -	3	0	$23\frac{173}{179}$.

7. A person is indebted to O 57*l* 15*s*, to P 108*l* 3*s* 8*d*, to Q 22*l* 10*d*, and to R 73*l*; but at his decease, his effects

* Contribution is a tax paid by provinces, towns, villages, &c. to excuse them from being plundered. It is paid in provisions or in money, and sometimes in both.

are

are found to be worth no more than 170*l* 14*s*; how must it be divided among his creditors?

Ans. o must have 37*l* 15*s* 5*d* $2\frac{3102}{10439}q$.
P - - - 70 15 2 $2\frac{7498}{10439}$.
Q - - - 14 8 4 $0\frac{4720}{10439}$.
R - - - 47 14 11 $2\frac{3318}{10439}$.

Ex. 8. A ship, worth 900*l*, being entirely lost, of which $\frac{1}{5}$ belonged to s, $\frac{1}{4}$ to T, and the rest to V; what loss will each sustain, supposing 540*l* of her were insured?

Ans. s will lose 45*l*, T 90*l*, and V 225*l*.

9. Four persons, W, X, Y, and Z, spent among them 25*s*, and agree that W shall pay $\frac{1}{2}$ of it, X $\frac{1}{3}$, Y $\frac{1}{4}$, and Z $\frac{1}{5}$; that is, their shares are to be in proportion as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$: what are their shares? Ans. W must pay 9*s* 8*d* $3\frac{41}{77}q$.
X - - 6 5 $3\frac{53}{77}$.
Y - - 4 10 $1\frac{58}{77}$.
Z - - 3 10 $3\frac{1}{77}$.

10. A detachment, consisting of 5 companies, being sent into a garrison, in which the duty required 76 men a day; what number of men must be furnished by each company, in proportion to their strength; the 1st consisting of 54 men, the 2d of 51 men, the 3d of 48 men, the 4th of 39, and the 5th of 36 men?

Ans. The 1st must furnish 18, the 2d 17, the 3d 16, the 4th 13, and the 5th 12 men*.

DOUBLE FELLOWSHIP.

Double Fellowship, as has been said, is concerned in cases in which the stocks of partners are employed or continued for different times.

* Questions of this nature frequently occurring in military service, General Haviland, an officer of great merit, contrived an ingenious instrument, for more expeditiously resolving them; which is distinguished by the name of the inventor, being called a Haviland.

Rule.

RULE*.—Multiply each person's stock by the time of its continuance; then divide the quantity, as in Single Fellowship, into shares, in proportion to these products, by saying,

As the total sum of all the said products,
Is to the whole gain or loss, or quantity to be parted,
So is each particular product,
To the correspondent share of the gain or loss.

EXAMPLES.

1. A had in company 50*l* for 4 months, and B had 60*l* for 5 months; at the end of which time they find 24*l* gained: how must it be divided between them?

$$\begin{array}{rccc} \text{Here} & 50 & & 60 \\ & 4 & & 5 \\ \hline & 200 & + & 300 = 500 \end{array}$$

Then, as 500 : 24 :: 200 : $9\frac{3}{5}$ = 9*l* 12*s* = A's share.
and as 500 : 24 :: 300 : $14\frac{2}{5}$ = 14 8 = B's share.

2. C and D hold a piece of ground in common, for which they are to pay 54*l*. C put in 23 horses for 27 days, and D 21 horses for 39 days; how much ought each man to pay of the rent?
Ans. C must pay 23*l* 5*s* 9*d*.
D must pay 30 14 3

3. Three persons, E, F, G, hold a pasture in common, for which they are to pay 30*l* per annum; into which E put 7 oxen for 3 months, F put 9 oxen for 5 months, and G put in 4 oxen for 12 months; how much must each person pay of the rent?
Ans. E must pay 5*l* 10*s* 6*d* $1\frac{5}{19}$*q*.
F - - 11 16 10 $0\frac{8}{19}$.
G - - 12 12 7 $2\frac{6}{19}$.

4. A ship's company take a prize of 1000*l*, which they agree to divide among them according to their pay and the time they have been on board: now the officers and midshipmen have been on board 6 months, and the sailors 3 months;

* The proof of this rule is as follows: When the times are equal, the shares of the gain or loss are evidently as the stocks, as in Single Fellowship; and when the stocks are equal, the shares are as the times; therefore, when neither are equal, the shares must be as their products.

the officers have 40*s* a month, the midshipmen 30*s*, and the sailors 22*s* a month; moreover there are 4 officers, 12 midshipmen, and 110 sailors: what will each man's share be?

Ans. each officer must have 23*l* 2*s* 5*d* $0\frac{92}{173}q$.
each midshipman - 17 6 9 $3\frac{69}{173}$.
each seaman - - 6 7 2 $0\frac{8}{173}$.

Ex. 5. H, with a capital of 1000*l*, began trade the first of January, and, meeting with success in business, took in I as a partner, with a capital of 1500*l*, on the first of March following. Three months after that they admit K as a third partner, who brought into stock 2800*l*. After trading together till the end of the year, they find there has been gained 1776*l* 10*s*; how must this be divided among the partners?

Ans. H must have 457*l* 9*s* $4\frac{1}{4}d$
I - - - 571 16 $8\frac{1}{4}$.
K - - - 747 3 $11\frac{1}{4}$.

6. X, Y, and Z made a joint-stock for 12 months; X at first put in 20*l*, and 4 months after 20*l* more; Y put in at first 30*l*, at the end of 3 months he put in 20*l* more, and 2 months after he put in 40*l* more; Z put in at first 60*l*, and 5 months after he put in 10*l* more, 1 month after which he took out 30*l*; during the 12 months they gained 50*l*; how much of it must each have?

Ans. X must have 10*l* 18*s* 6*d* $3\frac{40}{61}q$.
Y - - 22 8 1 $0\frac{21}{61}$.
Z - - 16 13 4 0.

SIMPLE INTEREST.

INTEREST is the premium or sum allowed for the loan, or forbearance of money. The money lent, or forborn, is called the Principal. And the sum of the principal and its interest, added together, is called the Amount. Interest is allowed at so much per cent. per annum; which premium per cent. per annum, or interest of 100*l* for a year, is called the rate of interest:—So,

When

When interest is at 3 per cent. the rate is 3;
- - - 4 per cent. - - 4;
- - - 5 per cent. - - 5;
- - - 6 per cent. - - 6;

But, by law, interest ought not to be taken higher than at the rate of 5 per cent.

Interest is of two sorts; Simple and Compound.

Simple Interest is that which is allowed for the principal lent or forborn only, for the whole time of forbearance. As the interest of any sum, for any time, is directly proportional to the principal sum, and also to the time of continuance; hence arises the following general rule of calculation.

As 100*l* is to the rate of interest, so is any given principal to its interest for one year. And again,

As 1 year is to any given time, so is the interest for a year, just found, to the interest of the given sum for that time.

OTHERWISE. Take the interest of 1 pound for a year, which multiply by the given principal, and this product again by the time of loan or forbearance, in years and parts, for the interest of the proposed sum for that time.

Note, When there are certain parts of years in the time, as quarters, or months, or days: they may be worked for, either by taking the aliquot or like parts of the interest of a year, or by the Rule of Three, in the usual way. Also, to divide by 100, is done by only pointing off two figures for decimals.

EXAMPLES.

1. To find the interest of 230*l* 10*s*, for 1 year, at the rate of 4 per cent. per annum.

Here, As 100 : 4 :: 230*l* 10*s* : 9*l* 4*s* 4$\frac{3}{4}$*d*.

```
              4
       ---------
100 )  9,22  0
         20
       -----
        4·40
         12
       -----
        4·80      Ans. 9l 4s 4¾d.
          4
       -----
        3·20
       -----
```

Ex. 2.

Ex. 2. To find the interest of 547*l* 15*s*, for 3 years, at 5 per cent. per annum.

As 100 : 5 :: 547·75 :
Or 20 : 1 :: 547·75 : 27·3875 interest for 1 year.
3
l 82·1625 ditto for 3 years.
20
s 3·2500
12
d 3·00 Ans. 82*l* 3*s* 3*d*.

3. To find the interest of 200 guineas, for 4 years 7 months and 25 days, at $4\frac{1}{2}$ per cent. per annum.

ds *l* ds
As 365 : 9·45 :: 25 : *l*
or 73 : 9·45 :: 5 : ·6472
5
73) 47·25 (·6472
345
530
19

210*l*
$4\frac{1}{2}$
840
105
9·45 interest for 1 yr.
4
37·80 ditto 4 years.
6 mo = $\frac{1}{2}$ 4·725 ditto 6 month.
1 mo = $\frac{1}{6}$ ·7875 ditto 1 month.
·6472 ditto 25 days.
l 43·9597
20
s 19·1940
12
d 2·3280
4 Ans. 43*l* 19*s* $2\frac{1}{4}$*d*.
q 1·3120

4. To find the interest of 450*l*, for a year, at 5 per cent. per annum. Ans. 22*l* 10*s*.

5. To find the interest of 715*l* 12*s* 6*d*, for a year, at $4\frac{1}{2}$ per cent. per annum. Ans. 32*l* 4*s* $0\frac{3}{4}$*d*.

6. To find the interest of 720*l*, for 3 years, at 5 per cent. per annum. Ans. 108*l*.

7. To find the interest of 355*l* 15*s* for 4 years, at 4 per cent. per annum. Ans. 56*l* 18*s* $4\frac{3}{4}$*d*.

Ex. 8.

Ex. 8. To find the interest of 32*l* 5*s* 8*d*, for 7 years, at $4\frac{1}{4}$ per cent. per annum. Ans. 9*l* 12*s* 1*d*.

9. To find the interest of 170*l*, for $1\frac{1}{2}$ year, at 5 per cent. per annum. Ans. 12*l* 15*s*.

10. To find the insurance on 205*l* 15*s*, for $\frac{1}{4}$ of a year, at 4 per cent. per annum. Ans. 2*l* 1*s* $1\frac{3}{4}$*d*.

11. To find the interest of 319*l* 6*d*, for $5\frac{3}{4}$ years, at $3\frac{3}{4}$ per cent. per annum. Ans. 68*l* 15*s* $9\frac{1}{2}$*d*.

12. To find the insurance on 107*l*, for 117 days, at $4\frac{3}{4}$ per cent. per annum. Ans. 1*l* 12*s* 7*d*.

13. To find the interest of 17*l* 5*s*, for 117 days, at $4\frac{3}{4}$ per cent. per annum. Ans. 5*s* 3*d*.

14. To find the insurance on 712*l* 6*s*, for 8 months, at $7\frac{1}{2}$ per cent. per annum. Ans. 35*l* 12*s* $3\frac{1}{2}$*d*.

Note. The Rules for Simple Interest, serve also to calculate Insurances, or the Purchase of Stocks, or any thing else that is rated at so much per cent.

· See also more on the subject of Interest, with the algebraical expression and investigation of the rules, at the end of the Algebra, next following.

COMPOUND INTEREST.

Compound Interest, called also Interest upon Interest, is that which arises from the principal and interest, taken together, as it becomes due, at the end of each stated time of payment. Though it be not lawful to lend money at Compound Interest, yet in purchasing annuities, pensions, or leases in reversion, it is usual to allow Compound Interest to the purchaser for his ready money.

Rules.—1. Find the amount of the given principal, for the time of the first payment, by Simple Interest. Then consider this amount as a new principal for the second payment, whose amount calculate as before. And so on through all the payments to the last, always accounting the last amount as a new principal for the next payment. The reason of which is evident from the definition of Compound Interest. *Or else*,

2. Find the amount of 1 pound for the time of the first payment, and raise or involve it to the power whose index is denoted by the number of payments. Then that power multiplied by the given principal, will produce the whole amount.

amount. From which the said principal being subtracted, leaves the Compound Interest of the same. As is evident from the first Rule.

EXAMPLES.

1. To find the amount of 720*l*, for 4 years, at 5 per cent. per annum.

Here 5 is the 20th part of 100, and the interest of 1*l* for a year is $\frac{1}{20}$ or ·05, and its amount 1·05. Therefore,

1. *By the 1st Rule.*

	l	*s*	*d*	
20)	720	0	0	1st yr's princip.
	36	0	0	1st yr's interest.
20)	756	0	0	2d yr's princip.
	37	16	0	2d yr's interest.
20)	793	16	0	3d yr's princip.
	39	13	$9\frac{1}{2}$	3d yr's interest.
20)	833	9	$9\frac{1}{2}$	4th yr's princip.
	41	13	$5\frac{1}{4}$	4th yr's interest
£	875	3	$3\frac{1}{4}$	the whole amo^t. or ans. required.

2. *By the 2d Rule.*

1·05	amount of 1*l*.
1·05	
1·1025	2d power of it.
1·1025	
1·21550625	4th pow. of it.
720	
l 875·1645	
20	
s 3·2900	
12	
d 3·4800	

2. To find the amount of 50*l*, in 5 years, at 5 per cent. per annum, compound interest. Ans. 63*l* 16*s* $3\frac{1}{4}$*d.*

3. To find the amount of 50*l* in 5 years, or 10 half-years, at 5 per cent. per annum, compound interest, the interest payable half-yearly. Ans. 64*l* 0*s* 1*d.*

4. To find the amount of 50*l*, in 5 years, or 20 quarters, at 5 per cent. per annum, compound interest, the interest payable quarterly. Ans. 64*l* 2*s* $0\frac{1}{4}$*d.*

5. To find the compound interest of 370*l* forborn for 6 years, at 4 per cent. per annum. Ans. 98*l* 3*s* $4\frac{1}{4}$*d.*

6. To find the compound interest of 410*l* forborn for $2\frac{1}{2}$ years, at $4\frac{1}{2}$ per cent. per annum, the interest payable half-yearly. Ans. 48*l* 4*s* $11\frac{1}{4}$*d.*

7. To find the amount, at compound interest, of 217*l*, forborn for $2\frac{1}{4}$ years, at 5 per cent. per annum, the interest payable quarterly. Ans. 242*l* 13*s* $4\frac{1}{2}$*d.*

Note. See the Rules for Compound Interest algebraically *investigated*, at the end of the Algebra.

ALLIGATION.

ALLIGATION.

Alligation teaches how to compound or mix together several simples of different qualities, so that the composition may be of some intermediate quality, or rate. It is commonly distinguished into two cases, Alligation Medial, and Alligation Alternate.

ALLIGATION MEDIAL.

Alligation Medial is the method of finding the rate or quality of the composition, from having the quantities and rates or qualities of the several simples given. And it is thus performed:

* Multiply the quantity of each ingredient by its rate or quality; then add all the products together, and add also all

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* *Demonstration.* The Rule is thus proved by Algebra.

Let a, b, c be the quantities of the ingredients;
and m, n, p their rates, or qualities, or prices;
then am, bn, cp are their several values,
and $am + bn + cp$ the sum of their values,
also $a + b + c$ is the sum of the quantities,
and if r denote the rate of the whole composition,

then $\overline{a + b + c} \times r$ will be the value of the whole,

conseq. $\overline{a + b + c} \times r = am + bn + cp$,

and $r = \overline{am + bn + cp} \div \overline{a + b + c}$, which is the Rule.

Note, If an ounce or any other quantity of pure gold be reduced into 24 equal parts, these parts are called Caracts; but gold is often mixed with some base metal, which is called the Alloy, and the mixture is said to be of so many caracts fine, according to the proportion of pure gold contained in it; thus, if 22 caracts of pure gold, and 2 of alloy be mixed together, it is said to be 22 caracts fine.

If any one of the simples be of little or no value with respect to the rest, its rate is supposed to be nothing; as water mixed with wine, and alloy with gold and silver.

the quantities together into another sum; then divide the former sum by the latter, that is, the sum of the products by the sum of the quantities, and the quotient will be the rate or quality of the composition required.

EXAMPLES.

1. If three sorts of gunpowder be mixed together, viz. 50lb at 12*d* a pound, 44lb at 9*d*, and 26lb at 8*d* a pound; how much a pound is the composition worth?

Here 50, 44, 26 are the quantities,
and 12, 9, 8 the rates or qualities;

$$\begin{array}{lrcl} \text{then} & 50 \times 12 & = & 600 \\ & 44 \times 9 & = & 396 \\ & 26 \times 8 & = & 208 \\ \hline & 120\,) & & 1204\,(10\tfrac{4}{120} = 10\tfrac{1}{30} \end{array}$$

Ans. The rate or price is $10\frac{1}{30}d$ the pound.

2. A composition being made of 5lb of tea at 7*s* per lb, 9lb at 8*s* 6*d* per lb, and $14\frac{1}{2}$lb at 5*s* 10*d* per lb; what is a lb of it worth? Ans. 6*s* $10\frac{1}{2}d$.

3. Mixed 4 gallons of wine at 4*s* 10*d* per gall, with 7 gallons at 5*s* 3*d* per gall, and $9\frac{3}{4}$ gallons at 5*s* 8*d* per gall.; what is a gallon of this composition worth? Ans. 5*s* $4\frac{1}{4}d$.

4. A mealman would mix 3 bushels of flour at 3*s* 5*d* per bushel, 4 bushels at 5*s* 6*d* per bushel, and 5 bushels at 4*s* 8*d* per bushel; what is the worth of a bushel of this mixture? Ans. 4*s* $7\frac{1}{2}d$.

5. A farmer mixes 10 bushels of wheat at 5*s* the bushel, with 18 bushels of rye at 3*s* the bushel, and 20 bushels of barley at 2*s* per bushel: how much is a bushel of the mixture worth? Ans. 3*s*.

6. Having melted together 7 oz of gold of 22 caracts fine, $12\frac{1}{2}$ oz of 21 caracts fine, and 17 oz of 19 caracts fine: I would know the fineness of the composition?

Ans. $20\frac{19}{73}$ caracts fine.

7. Of what fineness is that composition, which is made by mixing 3lb of silver of 9 oz fine, with 5lb 8 oz of 10 oz fine, and 1lb 10 oz of alloy. Ans. $7\frac{61}{64}$ oz fine.

ALLIGATION

ALLIGATION ALTERNATE.

ALLIGATION ALTERNATE is the method of finding what quantity of any number of simples, whose rates are given, will compose a mixture of a given rate. So that it is the reverse of Alligation Medial, and may be proved by it.

RULE I*.

1. SET the rates of the simples in a column under each other.—2. Connect, or link with a continued line, the rate of each simple, which is less than that of the compound, with one, or any number, of those that are greater than the compound; and each greater rate with one or any number of the less.—3. Write the difference between the mixture rate, and that of each of the simples, opposite the rate with which they are linked.—4. Then if only one difference stand against any rate, it will be the quantity belonging to that rate; but if there be several, their sum will be the quantity.

The examples may be proved by the rule for Alligation Medial.

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* *Demonst.* By connecting the less rate to the greater, and placing the difference between them and the rate alternately, the quantities resulting are such, that there is precisely as much gained by one quantity as is lost by the other, and therefore the gain and loss upon the whole is equal, and is exactly the proposed rate: and the same will be true of any other two simples managed according to the Rule.

In like manner, whatever the number of simples may be, and with how many soever every one is linked, since it is always a less with a greater than the mean price, there will be an equal balance of loss and gain between every two, and consequently an equal balance on the whole. Q. E. D.

It is obvious, from this Rule, that questions of this sort admit of a great variety of answers; for, having found one answer, we may find as many more as we please, by only multiplying or dividing each of the quantities found, by 2, or 3, or 4, &c: the reason of which is evident: for, if two quantities, of two simples, make a balance of loss and gain, with respect to the mean price, so must also the double or treble, the $\frac{1}{2}$ or $\frac{1}{3}$ part, or any other ratio of these quantities, and so on *ad infinitum*.

These kinds of questions are called by algebraists *indeterminate* or *unlimited* problems; and by an analytical process, theorems may be raised that will give all the *possible* answers.

EXAMPLES.

1. A merchant would mix wines at 16*s*, at 18*s*, and at 22*s* per gallon, so as that the mixture may be worth 20*s* the gallon: what quantity of each must be taken?

$$\text{Here } 20\left\{\begin{array}{l}16\\18\\22\end{array}\right. \quad \begin{array}{l}2 \text{ at } 16s\\2 \text{ at } 18s\\4+2=6 \text{ at } 22s.\end{array}$$

Ans. 2 gallons at 16*s*, 2 gallons at 18*s*, and 5 at 22*s*.

2. How much wine at 6*s* per gallon, and at 4*s* per gallon, must be mixed together, that the composition may be worth 5*s* per gallon? Ans. 1 qt, or 1 gall, &c.

3. How much sugar at 4*d*, at 6*d*, and at 11*d* per lb, must be mixed together, so that the composition formed by them may be worth 7*d* per lb?

Ans. 1 lb, or 1 stone, or 1 cwt, or any other equal quantity of each sort.

4. How much corn at 2*s* 6*d*, 3*s* 8*d*, 4*s*, and 4*s* 8*d* per bushel, must be mixed together, that the compound may be worth 3*s* 10*d* per bushel?

Ans. 2 at 2*s* 6*d*, 2 at 3*s* 8*d*, 3 at 4*s*, and 3 at 4*s* 8*d*.

5. A goldsmith has gold of 16, of 18, of 23, and of 24 caracts fine: how much must he take of each, to make it 21 caracts fine? Ans. 3 of 16, 2 of 18, 3 of 23, and 5 of 24.

6. It is required to mix brandy at 12*s*, wine at 10*s*, cyder at 1*s*, and water at 0 per gallon together, so that the mixture may be worth 8*s* per gallon?

Ans. 8 gals of brandy, 7 of wine, 2 of cyder, and 4 of water.

RULE II.

When the whole composition is limited to a certain quantity: Find an answer as before by linking; then say, as the sum of the quantities, or differences thus determined, is to the given quantity; so is each ingredient, found by linking, to the required quantity of each.

EXAMPLES.

1. How much gold of 15, 17, 18, and 22 caracts fine, must be mixed together, to form a composition of 40 oz of 20 caracts fine?

Here

$$\text{Here } 20\left\{\begin{array}{l}15\\17\\18\\22\end{array}\right. \quad \begin{array}{r} 2\\ 2\\ 2\\ 5+3+2=10\\ \hline 16\end{array}$$

Then, as 16 : 40 :: 2 : 5
and 16 : 40 :: 10 : 25

Ans. 5 oz of 15, of 17, and of 18 caracts fine, and 25 oz of 22 caracts fine*.

Ex. 2. A vintner has wine at 4*s*, at 5*s*, at 5*s* 6*d*, and at 6*s* a gallon; and he would make a mixture of 18 gallons, so that it might be afforded at 5*s* 4*d* per gallon; how much of each sort must he take?

Ans. 3 gal. at 4*s*, 3 at 5*s*, 6 at 5*s* 6*d*, and 6 at 6*s*.

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* A great number of questions might be here given relating to the specific gravities of metals, &c. but one of the most curious may here suffice.

Hiero, king of Syracuse, gave orders for a crown to be made entirely of pure gold; but suspecting the workman had debased it by mixing it with silver or copper, he recommended the discovery of the fraud to the famous Archimedes, and desired to know the exact quantity of alloy in the crown.

Archimedes, in order to detect the imposition, procured two other masses, the one of pure gold, the other of silver or copper, and each of the same weight with the former; and by putting each separately into a vessel full of water, the quantity of water expelled by them determined their specific gravities; from which, and their given weights, the exact quantities of gold and alloy in the crown may be determined.

Suppose the weight of each crown to be 10lb, and that the water expelled by the copper or silver was 92lb, by the gold 52lb, and by the compound crown 64lb; what will be the quantities of gold and alloy in the crown?

The rates of the simples are 92 and 52, and of the compound 64; therefore

$$64 \left| \begin{array}{l} 92 \\ 52 \end{array} \right. \begin{array}{l} 12 \text{ of copper} \\ 28 \text{ of gold} \end{array}$$

And the sum of these is 12 + 28 = 40, which should have been but 10; therefore by the Rule,

$$\left.\begin{array}{l} 40 : 10 :: 12 : 3\text{lb of copper} \\ 40 : 10 :: 28 : 7\text{lb of gold} \end{array}\right\} \text{the answer.}$$

RULE

RULE III*.

When one of the ingredients is limited to a certain quantity; Take the difference between each price, and the mean rate as before; then say, As the difference of that simple, whose quantity is given, is to the rest of the differences severally; so is the quantity given, to the several quantities required.

EXAMPLES.

1. How much wine at 5*s*, at 5*s* 6*d*, and 6*s* the gallon, must be mixed with 3 gallons at 4*s* per gallon, so that the mixture may be worth 5*s* 4*d* per gallon?

$$\text{Here } 64 \left\{ \begin{array}{l} 48 \\ 60 \\ 66 \\ 72 \end{array} \right. \qquad \begin{array}{rcccl} 8 & + & 2 & = & 10 \\ 8 & + & 2 & = & 10 \\ 16 & + & 4 & = & 20 \\ 16 & + & 4 & = & 20 \end{array}$$

$$\begin{array}{ll} \text{Then} & 10 : 10 :: 3 : 3 \\ & 10 : 20 :: 3 : 6 \\ & 10 : 20 :: 3 : 6 \end{array}$$

Ans. 3 gallons at 5*s*, 6 at 5*s* 6*d*, and 6 at 6*s*.

2. A grocer would mix teas at 12*s*, 10*s*, and 6*s* per lb, with 20lb at 4*s* per lb. how much of each sort must he take to make the composition worth 8*s* per lb?

Ans. 20lb at 4*s*, 10lb at 6*s*, 10lb at 10*s*, and 20lb at 12*s*.

3. How much gold of 15, of 17, and of 22 caracts fine, must be mixed with 5 oz of 18 caracts fine, so that the composition may be 20 caracts fine?

Ans. 5 oz. of 15 caracts fine, 5 oz of 17, and 25 of 22.

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* In the very same manner questions may be wrought when several of the ingredients are limited to certain quantities, by finding first for one limit, and then for another. The two last Rules can need no demonstration, as they evidently result from the first, the reason of which has been already explained.

POSITION.

POSITION is a method of performing certain questions, which cannot be resolved by the common direct rules. It is sometimes called False Position, or False Supposition, because it makes a supposition of false numbers, to work with the same as if they were the true ones, and by their means discovers the true numbers sought. It is sometimes also called Trial-and-Error, because it proceeds by *trials* of false numbers, and thence finds out the true ones by a comparison of the *errors*.—Position is either Single or Double.

SINGLE POSITION.

SINGLE POSITION is that by which a question is resolved by means of one supposition only. Questions which have their result proportional to their suppositions, belong to Single Position: such as those which require the multiplication or division of the number sought by any proposed number; or when it is to be increased or diminished by itself, or any parts of itself, a certain proposed number of times. The rule is as follows:

TAKE or assume any number for that which is required, and perform the same operations with it, as are described or performed in the question. Then say, As the result of the said operation, is to the position, or number assumed; so is the result in the question, to a fourth term, which will be the number sought *.

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* The reason of this Rule is evident, because it is supposed that the results are proportional to the suppositions.

Thus, $na : a :: nz : z$,

or $\frac{a}{n} : a :: \frac{z}{n} : z$,

or $\frac{a}{n} \pm \frac{a}{m}$ &c $: a :: \frac{z}{n} \pm \frac{z}{m}$ &c $: z$,

and so on.

EXAMPLES.

EXAMPLES.

1. A person after spending $\frac{1}{3}$ and $\frac{1}{4}$ of his money, has yet remaining 60*l*; what had he at first?

Suppose he had at first 120*l*.	Proof.
Now $\frac{1}{3}$ of 120 is 40	$\frac{1}{3}$ of 144 is 48
$\frac{1}{4}$ of it is 30	$\frac{1}{4}$ of 144 is 36
their sum is 70	their sum 84
which taken from 120	taken from 144
leaves 50	leaves 60 as
Then, 50 : 120 :: 60 : 144, the Answer.	per question.

2. What number is that, which being multipied by 7, and the product divided by 6, the quotient may be 21? Ans. 18.

3. What number is that, which being increased by $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ of itself, the sum shall be 75? Ans. 36.

4. A general, after sending out a foraging $\frac{1}{2}$ and $\frac{1}{3}$ of his men, had yet remaining 1000: what number had he in command? Ans. 6000.

5. A gentleman distributed 52 pence among a number of poor people, consisting of men, women, and children; to each man he gave 6*d*, to each woman 4*d*, and to each child 2*d*: moreover there were twice as many women as men, and thrice as many children as women. How many were there of each? Ans. 2 men, 4 women, and 12 children.

6. One being asked his age, said, if $\frac{3}{5}$ of the years I have lived, be multiplied by 7, and $\frac{2}{3}$ of them be added to the product, the sum will be 219. What was his age?

Ans. 45 years.

DOUBLE

DOUBLE POSITION.

Double Position is the method of resolving certain questions by means of two suppositions of false numbers.

To the Double Rule of Position belong such questions as have their results not proportional to their positions: such are those, in which the numbers sought, or their parts, or their multiples, are increased or diminished by some given absolute number, which is no known part of the number sought.

RULE I*.

Take or assume any two convenient numbers, and proceed with each of them separately, according to the conditions of the question, as in Single Position; and find how much each result is different from the result mentioned in the question, calling these differences the *errors*, noting also whether the results are too great or too little.

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* *Demonstr.* The Rule is founded on this supposition, namely, that the first error is to the second, as the difference between the true and first supposed number, is to the difference between the true and second supposed number; when that is not the case, the exact answer to the question cannot be found by this Rule.—That the Rule is true, according to that supposition, may be thus proved.

Let a and b be the two suppositions, and A and B their results, produced by similar operation; also r and s their errors, or the differences between the results A and B from the true result N; and let x denote the number sought, answering to the true result N of the question.

Then is $N - A = r$, and $N - B = s$. And, according to the supposition on which the Rule is founded, $r : s :: x-a : x-b$; hence, by multiplying extremes and means, $rx - rb = sx - sa$; then, by transposition, $rx - sx = rb - sa$; and, by division, $x = \frac{rb - sa}{r - s}$ = the number sought, which is the rule when the results are both too little.

If the results be both too great, so that A and B are both greater than N; then $N - A = -r$, and $N - B = -s$, or r and s are both negative; hence $-r : -s :: x - a : x - b$, but $-r : -s :: +r : +s$, therefore $r : s :: x-a : x - b$; and the rest will be exactly as in the former case.

But if one result A only be too little, and the other B too great, or one error r positive, and the other s negative, then the theorem becomes $x = \frac{rb + sa}{r + s}$, which is the Rule in this case, or when the errors are unlike.

Then

Then multiply each of the said errors by the contrary supposition, namely, the first position by the second error, and the second position by the first error. Then,

If the errors are alike, divide the difference of the products by the difference of the errors, and the quotient will be the answer.

But if the errors are unlike, divide the sum of the products by the sum of the errors, for the answer.

Note, The errors are said to be alike, when they are either both too great or both too little; and unlike, when one is too great and the other too little.

EXAMPLES.

1. What number is that, which being multiplied by 6, the product increased by 18, and the sum divided by 9, the quotient shall be 20?

Suppose the two numbers 18 and 30. Then,

	First Position.		Second Position.		Proof.
	18	Suppose	30		27
	6	mult.	6		6
	108		180		162
	18	add	18		18
9)	126	div.	9) 198		9) 180
	14	results	22		20
	20	true res.	20		
	+6	errors unlike	−2		
2d pos.	30	mult.	18	1st pos.	
Er- { 2	180		36		
rors { 6	36				
sum 8)	216	sum of products			
	27	Answer sought.			

RULE II.

Find, by trial, two numbers, as near the true number as convenient, and work with them as in the question; marking the errors which arise from each of them.

Multiply the difference of the two numbers assumed, or found by trial, by one of the errors, and divide the product by the difference of the errors, when they are alike, but by their sum when they are unlike.

Add the quotient, last found, to the number belonging to the said error, when that number is too little, but subtract it

it when too great, and the result will give the true quantity sought*.

EXAMPLES.

1. So, the foregoing example, worked by this 2d rule, will be as follows:

30 positions 18;		their dif.	12
−2 errors + 6;		least error	2
	sum of errors 8)	24 (3 subtr.	
	from the position	30	
	leaves the answer	27	

Ex. 2. A son asking his father how old he was, received this answer: Your age is now one-third of mine; but 5 years ago, your age was only one-fourth of mine. What then are their two ages? Ans. 15 and 45.

3. A workman was hired for 20 days, at 3*s* per day, for every day he worked; but with this condition, that for every day he played, he should forfeit 1*s*. Now it so happened, that upon the whole he had 2*l* 4*s* to receive. How many of the days did he work? Ans. 16.

4. A and B began to play together with equal sums of money: A first won 20 guineas, but afterwards lost back $\frac{2}{3}$ of what he then had; after which, B had 4 times as much as A. What sum did each begin with? Ans. 100 guineas.

5. Two persons, A and B, have both the same income, A saves $\frac{1}{5}$ of his; but B, by spending 50*l* per annum more than A, at the end of 4 years finds himself 100*l* in debt. What does each receive and spend per annum?

Ans. They receive 125*l* per annum; also A spends 100*l*, and B spends 150*l* per annum.

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* For since, by the supposition, $r : s :: x - a : x - b$, therefore by division, $r - s : s :: b - a : x - b$, which is the 2d Rule.

PRACTICAL

PRACTICAL QUESTIONS IN ARITHMETIC.

QUEST. 1. The swiftest velocity of a cannon-ball, is about 2000 feet in a second of time. Then in what time, at that rate, would such a ball be in moving from the earth to the sun, admitting the distance to be 100 millions of miles, and the year to contain 365 days 6 hours?

Ans. $8\frac{4308}{13149}$ years.

QUEST. 2. What is the ratio of the velocity of light to that of a cannon-ball, which issues from the gun with a velocity of 1500 feet per second; light passing from the sun to the earth in $7\frac{1}{2}$ minutes? Ans. the ratio of $782222\frac{2}{9}$ to 1.

QUEST. 3. The slow or parade-step being 70 paces per minute, at 28 inches each pace, it is required to determine at what rate per hour that movement is? Ans. $1\frac{113}{132}$ miles.

QUEST. 4. The quick-time or step, in marching, being 2 paces per second, or 120 per minute, at 28 inches each; then at what rate per hour does a troop march on a route, and how long will they be in arriving at a garrison 20 miles distant, allowing a halt of one hour by the way to refresh?

Ans. $\begin{cases} \text{the rate is } 3\frac{2}{11} \text{ miles an hour.} \\ \text{and the time } 7\frac{2}{7} \text{ hr, or 7 h } 17\frac{1}{7} \text{ min.} \end{cases}$

QUEST. 5. A wall was to be built 700 yards long in 29 days. Now, after 12 men had been employed on it for 11 days, it was found that they had completed only 220 yards of the wall. It is required then to determine how many men must be added to the former, that the whole number of them may just finish the wall in the time proposed, at the same rate of working. Ans. 4 men to be added.

QUEST. 6. To determine how far 500 millions of guineas will reach, when laid down in a straight line touching one another; supposing each guinea to be an inch in diameter, as it is very nearly. Ans. 7891 miles, 728 yds, 2 ft, 8 in.

QUEST. 7. Two persons, A and B, being on opposite sides of a wood, which is 536 yards about, they begin to go round it, both the same way, at the same instant of time; A goes at the rate of 11 yards per minute, and B 34 yards in 3 minutes; the question is, how many times will the wood be gone round before the quicker overtake the slower?

Ans. 17 times.

QUEST.

QUEST. 8. A can do a piece of work alone in 12 days, and B alone in 14; in what time will they both together perform a like quantity of work? Ans. $6\frac{6}{13}$ days.

QUEST. 9. A person who was possessed of a $\frac{3}{5}$ share of a copper mine, sold $\frac{3}{4}$ of his interest in it for 1800*l*; what was the reputed value of the whole at the same rate? Ans. 4000*l*.

QUEST. 10. A person after spending 20*l* more than $\frac{1}{4}$ of his yearly income, had then remaining 30*l* more than the half of it; what was his income? Ans. 200*l*.

QUEST. 11. The hour and minute hand of a clock are exactly together at 12 o'clock; when are they next together? Ans. at $1\frac{1}{11}$ hr, or 1 hr, $5\frac{5}{11}$ min.

QUEST. 12. If a gentleman whose annual income is 1500*l*, spend 20 guineas a week; whether will he save or run in debt, and how much in the year? Ans. save 408*l*.

QUEST. 13. A person bought 180 oranges at 2 a penny, and 180 more at 3 a penny; after which, selling them out again at 5 for 2 pence, whether did he gain or lose by the bargain? Ans. he lost 6 pence.

QUEST. 14. If a quantity of provisions serves 1500 men 12 weeks, at the rate of 20 ounces a day for each man; how many men will the same provisions maintain for 20 weeks, at the rate of 8 ounces a day for each man? Ans. 2250 men.

QUEST. 15. In the latitude of London, the distance round the earth, measured on the parallel of latitude, is about 15550 miles; now as the earth turns round in 23 hours 56 minutes, at what rate per hour is the city of London carried by this motion from west to east? Ans. $649\frac{259}{359}$ miles an hour.

QUEST. 16. A father left his son a fortune, $\frac{1}{4}$ of which he ran through in 8 months: $\frac{3}{7}$ of the remainder lasted him 12 months longer; after which he had bare 820*l* left. What sum did the father bequeath his son? Ans. 1913*l* 6*s* 8*d*.

QUEST. 17. If 1000 men, besieged in a town, with provisions for 5 weeks, allowing each man 16 ounces a day, be reinforced with 500 men more; and supposing that they cannot be relieved till the end of 8 weeks, how many ounces a day must each man have, that the provision may last that time? Ans. $6\frac{2}{3}$ ounces.

QUEST. 18. A younger brother received 8400*l*, which was just $\frac{7}{9}$ of his elder brother's fortune: What was the father worth at his death? Ans. 19200*l*.

QUEST.

QUEST. 19. A person, looking on his watch, was asked what was the time of the day, who answered, It is between 5 and 6; but a more particular answer being required, he said that the hour and minute hands were then exactly together: What was the time? Ans. $27\frac{3}{11}$ min. past 5.

QUEST. 20. If 20 men can perform a piece of work in 12 days, how many men will accomplish another thrice as large in one-fifth of the time? Ans. 300.

QUEST. 21. A father devised $\frac{7}{18}$ of his estate to one of his sons, and $\frac{7}{18}$ of the residue to another, and the surplus to his relict for life. The children's legacies were found to be 514*l* 6*s* 8*d* different: Then what money did he leave the widow the use of? Ans. 1270*l* 1*s* $9\frac{11}{49}$*d*.

QUEST. 22. A person, making his will, gave to one child $\frac{13}{20}$ of his estate, and the rest to another. When these legacies came to be paid the one turned out 1200*l* more than the other: What did the testator die worth? Ans. 4000*l*.

QUEST. 23. Two persons, A and B, travel between London and Lincoln, distant 100 miles, A from London, and B from Lincoln, at the same instant. After 7 hours they meet on the road, when it appeared that A had rode $1\frac{1}{2}$ miles an hour more than B. At what rate per hour then did each of the travellers ride? Ans. A $7\frac{25}{28}$, and B $6\frac{11}{28}$ miles.

QUEST. 24. Two persons, A and B, travel between London and Exeter. A leaves Exeter at 8 o'clock in the morning, and walks at the rate of 3 miles an hour, without intermission; and B sets out from London at 4 o'clock the same evening, and walks for Exeter at the rate of 4 miles an hour constantly. Now, supposing the distance between the two cities to be 130 miles, whereabouts on the road will they meet? Ans. $69\frac{3}{7}$ miles from Exeter.

QUEST. 25. One hundred eggs being placed on the ground, in a straight line, at the distance of a yard from each other: How far will a person travel who shall bring them one by one to a basket, which is placed at one yard from the first egg? Ans. 10100 yards, or 5 miles and 1300 yds.

QUEST. 26. The clocks of Italy go on to 24 hours: Then how many strokes do they strike in one complete revolution of the index? Ans. 300.

QUEST. 27. One Sessa, an Indian, having invented the game of chess, shewed it to his prince, who was so delighted with

with it, that he promised him any reward he should ask; on which Sessa requested that he might be allowed one grain of wheat for the first square on the chess board, 2 for the second, 4 for the third, and so on, doubling continually, to 64, the whole number of squares. Now, supposing a pint to contain 7680 of these grains, and one quarter or 8 bushels to be worth 27*s* 6*d*, it is required to compute the value of all the corn?

Ans. 6450468216285*l* 17*s* 3*d* $3\frac{32757}{32768}$*q*.

QUEST. 28. A person increased his estate annually by 100*l* more than the $\frac{1}{4}$ part of it; and at the end of 4 years found that his estate amounted to 10342*l* 3*s* 9*d*. What had he at first? Ans. 4000*l*.

QUEST. 29. Paid 1012*l* 10*s* for a principal of 750*l*, taken in 7 years before: at what rate per cent. per annum did I pay interest? Ans. 5 per cent.

QUEST. 30. Divide 1000*l* among A, B, C; so as to give A 120 more, and B 95 less than C.

Ans. A 445, B 230, C 325.

QUEST. 31. A person being asked the hour of the day, said, the time past noon is equal to $\frac{4}{5}$ths of the time till midnight. What was the time? Ans. 20 min. past 5.

QUEST. 32. Suppose that I have $\frac{3}{16}$ of a ship worth 1200*l*; what part of her have I left after selling $\frac{2}{5}$ of $\frac{4}{9}$ of my share, and what is it worth? Ans. $\frac{37}{240}$, worth 185*l*.

QUEST. 33. Part 1200 acres of land among A, B, C; so that B may have 100 more than A, and C 64 more than B.

Ans. A 312, B 412, C 476.

QUEST. 34. What number is that, from which if there be taken $\frac{2}{7}$ of $\frac{3}{8}$, and to the remainder be added $\frac{9}{15}$ of $\frac{5}{16}$, the sum will be 10? Ans. $9\frac{79}{84}$.

QUEST. 35. There is a number which, if multiplied by $\frac{2}{3}$ of $\frac{4}{5}$ of $1\frac{1}{2}$, will produce 1: what is the square of that number? Ans. $1\frac{9}{16}$.

QUEST. 36. What length must be cut off a board, $8\frac{1}{2}$ inches broad, to contain a square foot, or as much as 12 inches in length and 12 in breadth? Ans. $16\frac{16}{17}$ inches.

QUEST. 37. What sum of money will amount to 138*l* 2*s* 6*d*, in 15 months, at 5 per cent. per annum simple interest?

Ans. 130*l*.

QUEST. 38. A father divided his fortune among his three sons, A, B, C, giving A 4 as often as B 3, and C 5 as often as

B 6;

B 6; what was the whole legacy, supposing A's share was 4000*l*. Ans. 9500*l*.

QUEST. 39. A young hare starts 40 yards before a greyhound, and is not perceived by him till she has been up 40 seconds; she scuds away at the rate of 10 miles an hour, and the dog, on view, makes after her at the rate of 18: how long will the course hold, and what ground will be run over, counting from the outsetting of the dog?

Ans. $60\frac{5}{12}$ sec. and 530 yards run.

QUEST. 40. Two young gentlemen, without private fortune, obtain commissions at the same time, and at the age of 18. One thoughtlessly spends 10*l* a year more than his pay; but, shocked at the idea of not paying his debts, gives his creditor a bond for the money, at the end of every year, and also insures his life for the amount; each bond costs him 30 shillings, besides the lawful interest of 5 per cent. and to insure his life costs him 6 per cent.

The other, having a proper pride, is determined never to run in debt; and, that he may assist a friend in need, perseveres in saving 10*l* every year, for which he obtains an interest of 5 per cent. which interest is every year added to his savings, and laid out, so as to answer the effect of compound interest.

Suppose these two officers to meet at the age of 50, when each receives from Government 400*l* per annum; that the one, seeing his past errors, is resolved in future to spend no more than he actually has, after paying the interest for what he owes, and the insurance on his life.

The other, having now something before hand, means in future, to spend his full income, without increasing his stock.

It is desirable to know how much each has to spend per annum, and what money the latter has by him to assist the distressed, or leave to those who deserve it?

Ans. The reformed officer has to spend 66*l* 19*s* $1\frac{1}{4}$·5389*d* per annum.

The prudent officer has to spend 437*l* 12*s* $11\frac{3}{4}$·4379*d* per annum.

And the latter has saved, to dispose of, 752*l* 19*s* 9·1896*d*.

END OF THE ARITHMETIC.

OF LOGARITHMS*.

LOGARITHMS are made to facilitate troublesome calculations in numbers. This they do, because they perform multiplication by only addition, and division by only subtraction, and raising of powers by multiplying the logarithm by the index of the power, and extracting of roots by dividing the logarithm of the number by the index of the root. For, logarithms are numbers so contrived, and adapted to other numbers, that the sums and differences of the former shall correspond to, and show, the products and quotients of the latter, &c.

Or, more generally, logarithms are the numerical exponents of ratios; or they are a series of numbers in arithmetical

* The invention of Logarithms is due to Lord Napier, Baron of Merchiston, in Scotland, and is properly considered as one of the most useful inventions of modern times. A table of these numbers was first published by the inventor at Edinburgh, in the year 1614, in a treatise entitled *Canon Mirificum Logarithmorum;* which was eagerly received by all the learned throughout Europe. Mr. Henry Briggs, then professor of geometry at Gresham College, soon after the discovery, went to visit the noble inventor; after which, they jointly undertook the arduous task of computing new tables on this subject, and reducing them to a more convenient form than that which was at first thought of. But Lord Napier dying soon after, the whole burden fell upon Mr. Briggs, who, with prodigious labour and great skill, made an entire Canon, according to the new form, for all numbers from 1 to 20000, and from 90000 to 10100, to 14 places of figures, and published it at London in the year 1624, in a treatise entitled Arithmetica Logarithmica, with directions for supplying the intermediate parts.

 This

metical progression, answering to another series of numbers in geometrical progression.

Thus	0,	1,	2,	3,	4,	5,	6,	Indices, or logarithms.
	1,	2,	4,	8,	16,	32,	64,	Geometric progression.
Or	0,	1,	2,	3,	4,	5,	6,	Indices, or logarithms.
	1,	3,	9,	27,	81,	243,	729,	Geometric progression.
Or	0,	1,	2,	3,	4,	5,		Indices, or logs.
	1,	10,	100,	1000,	10000,	100000,		Geom. progres.

Where it is evident, that the same indices serve equally for any geometric series; and consequently there may be an

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This Canon was again published in Holland by Adrian Vlacq, in the year 1628, together with the Logarithms of all the numbers which Mr. Briggs had omitted; but he contracted them down to 10 places of decimals. Mr. Briggs also computed the Logarithms of the sines, tangents, and secants, to every degree, and centesm, or 100th part of a degree, of the whole quadrant; and annexed them to the natural sines, tangents, and secants, which he had before computed, to fifteen places of figures. These Tables, with their construction and use, were first published in the year 1633, after Mr. Briggs's death, by Mr. Henry Gellibrand, under the title of Trigonometria Britannica.

Benjamin Ursinus also gave a Table of Napier's Logs. and of sines, to every 10 seconds. And Chr. Wolf, in his Mathematical Lexicon, says that one Van Loser had computed them to every single second, but his untimely death prevented their publication. Many other authors have treated on this subject; but as their numbers are frequently inaccurate and incommodiously disposed, they are now generally neglected. The Tables in most repute at present, are those of Gardiner in 4to, first published in the year 1742; and my own Tables in 8vo, first printed in the year 1785, where the Logarithms of all numbers may be easily found from 1 to 10000000; and those of the sines, tangents, and secants, to any degree of accuracy required.

Also, Mr. Michael Taylor's Tables in large 4to, containing the common logarithms, and the logarithmic sines and tangents to every second of the quadrant. And, in France, the new book of logarithms by Callet; the 2d edition of which, in 1795, has the tables still farther extended, and are printed with what are called stereotypes, the types in each page being soldered together into a solid mass or block.

Dodson's Antilogarithmic Canon is likewise a very elaborate work, and used for finding the numbers answering to any given logarithm

endless variety of systems of logarithms, to the same common numbers, by only changing the second term, 2, 3, or 10, &c. of the geometrical series of whole numbers; and by interpolation the whole system of numbers may be made to enter the geometric series, and receive their proportional logarithms, whether integers or decimals.

It is also apparent, from the nature of these series, that if any two indices be added together, their sum will be the index of that number which is equal to the product of the two terms, in the geometric progression, to which those indices belong. Thus, the indices 2 and 3, being added together, make 5; and the numbers 4 and 8, or the terms corresponding to those indices, being multiplied together, make 32, which is the number answering to the index 5.

In like manner, if any one index be subtracted from another, the difference will be the index of that number which is equal to the quotient of the two terms to which those indices belong. Thus, the index 6, minus the index 4, is = 2; and the terms corresponding to those indices are 64 and 16, whose quotient is = 4, which is the number answering to the index 2.

For the same reason, if the logarithm of any number be multiplied by the index of its power, the product will be equal to the logarithm of that power. Thus, the index or logarithm of 4, in the above series, is 2; and if this number be multiplied by 3, the product will be = 6; which is the logarithm of 64, or the third power of 4.

And, if the logarithm of any number be divided by the index of its root, the quotient will be equal to the logarithm of that root. Thus, the index or logarithm of 64 is 6; and if this number be divided by 2, the quotient will be = 3; which is the logarithm of 8, or the square root of 64.

The logarithms most convenient for practice, are such as are adapted to a geometric series increasing in a tenfold proportion, as in the last of the above forms; and are those which are to be found, at present, in most of the common tables on this subject. The distinguishing mark of this system of logarithms is, that the index or logarithm of 10 is 1; that of 100 is 2; that of 1000 is 3; &c. And, in

 decimals,

decimals, the logarithm of ·1 is −1; that of ·01 is −2; that of ·001 is −3; &c. The log. of 1 being 0 in every system. Whence it follows, that the logarithm of any number between 1 and 10, must be 0 and some fractional parts; and that of a number between 10 and 100, will be 1 and some fractional parts; and so on, for any other number whatever. And since the integral part of a logarithm, usually called the Index, or Characteristic, is always thus readily found, it is commonly omitted in the tables; being left to be supplied by the operator himself, as occasion requires.

Another Definition of Logarithms is, that the logarithm of any number is the index of that power of some other number, which is equal to the given number. So, if there be $N = r^n$, then n is the log. of N; where n may be either positive or negative, or nothing, and the root r any number whatever, according to the different systems of logarithms. When n is $= 0$, then N is $= 1$, whatever the value of r is; which shows, that the log. of 1 is always 0, in every system of logarithms. When n is $= 1$. then N is $= r$; so that the radix r is always that number whose log. is 1, in every system. When the radix r is $= 2{\cdot}718281828459$ &c, the indices n are the hyperbolic or Napier's log. of the numbers N; so that n is always the hyp. log. of the number N or $(2{\cdot}718\ \&c.)^n$.

But when the radix r is $= 10$, then the index n becomes the common or Briggs's log. of the number N: so that the common log. of any number 10^n or N, is n the index of that power of 10 which is equal to the said number. Thus 100, being the second power of 10, will have 2 for its logarithm; and 1000, being the third power of 10, will have 3 for its logarithm: hence also, if 50 be $= 10^{1{\cdot}69897}$, then is $1{\cdot}69897$ the common log. of 50. And, in general, the following decuple series of terms,

viz. 10^4,	10^3,	10^2,	10^1,	10^0,	10^{-1},	10^{-2},	10^{-3},	10^{-4},
or 10000,	1000,	100,	10,	1,	·1,	·01,	·001,	·0001,
have 4,	3,	2,	1,	0,	−1,	−2,	−3,	−4,

for their logarithms, respectively. And from this scale of numbers and logarithms, the same properties easily follow, as above mentioned.

PROBLEM.

PROBLEM.

To compute the Logarithm to any of the Natural Numbers 1, 2, 3, 4, 5, &c.

RULE I*.

TAKE the geometric series, 1, 10, 100, 1000, 10000, &c. and apply to it the arithmetic series, 0, 1, 2, 3, 4, &c. as logarithms.—Find a geometric mean between 1 and 10, or between 10 and 100, or any other two adjacent terms of the series, between which the number proposed lies.—In like manner, between the mean, thus found, and the nearest extreme, find another geometrical mean; and so on, till you arrive within the proposed limit of the number whose logarithm is sought.—Find also as many arithmetical means, in the same order as you found the geometrical ones, and these will be the logarithms answering to the said geometrical means.

EXAMPLE.

Let it be required to find the logarithm of 9.

Here the proposed number lies between 1 and 10.

First, then, the log. of 10 is 1, and the log. of 1 is 0;
theref. $\overline{1 + 0} \div 2 = \frac{1}{2} = \cdot 5$ is the arithmetical mean,
and $\sqrt{10 \times 1} = \sqrt{10} = 3{\cdot}1622777$ the geom. mean;
hence the log. of 3·1622777 is ·5.

Secondly, the log. of 10 is 1, and the log. of 3·1622777 is ·5;
theref. $\overline{1 + \cdot 5} \div 2 = \cdot 75$ is the arithmetical mean,
and $\sqrt{10 \times 3{\cdot}1622777} = 5{\cdot}6234132$ is the geom. mean;
hence the log. of 5·6234132 is ·75.

Thirdly, the log. of 10 is 1, and the log. of 5·6234132 is ·75;
theref. $\overline{1 + \cdot 75} \div 2 = \cdot 875$ is the arithmetical mean,
and $\sqrt{10 \times 5{\cdot}6235132} = 7{\cdot}4989422$ the geom. mean;
hence the log. of 7·4989422 is ·875.

Fourthly, the log. of 10 is 1, and the log. of 7·4989422 is ·875;
theref. $\overline{1 + \cdot 875} \div 2 = \cdot 9375$ is the arithmetical mean,
and $\sqrt{10 \times 7{\cdot}4989422} = 8{\cdot}6596431$ the geom. mean;
hence the log. of 8·6596431 is ·9375.

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* The reader who wishes to inform himself more particularly concerning the history, nature, and construction of Logarithms, may consult the Introduction to my Mathematical Tables, lately published, where he will find his curiosity amply gratified.

Fifthly,

Fifthly, the log. of 10 is 1, and the log. of 8·6596431 is ·9375;

theref. $\overline{1 + \cdot 9375} \div 2 = \cdot 96875$ is the arithmetical mean,

and $\sqrt{10 \times 8\cdot 6596431} = 9\cdot 3057204$ the geom. mean;

hence the log. of 9·3057204 is ·96875.

Sixthly, the log. of 8·6596431 is ·9375, and the log. of 9·3057204 is ·96875;

theref. $\overline{\cdot 9375 + \cdot 96875} \div 2 = \cdot 953125$ is the arith. mean,

and $\sqrt{8\cdot 6596431 \times 9\cdot 3057204} = 8\cdot 9768713$ the geometric mean;

hence the log. of 8·9768713 is ·953125.

And proceeding in this manner, after 25 extractions, it will be found that the logarithm of 8·9999998 is ·9542425; which may be taken for the logarithm of 9, as it differs so little from it, that it is sufficiently exact for all practical purposes. And in this manner were the logarithms of almost all the prime numbers at first computed.

RULE II*.

Let b be the number whose logarithm is required to be found; and a the number next less than b, so that $b - a = 1$, the logarithm of a being known; and let s denote the sum of the two numbers $a + b$. Then

1. Divide the constant decimal ·8685889638 &c, by s, and reserve the quotient: divide the reserved quotient by the square of s, and reserve this quotient: divide this last quotient also by the square of s, and again reserve the quotient: and thus proceed, continually dividing the last quotient by the square of s, as long as division can be made.

2. Then write these quotients orderly under one another, the first uppermost, and divide them respectively by the odd numbers, 1, 3, 5, 7, 9, &c, as long as division can be made; that is, divide the first reserved quotient by 1, the second by 3, the third by 5, the fourth by 7, and so on.

3. Add all these last quotients together, and the sum will be the logarithm of $b \div a$; therefore to this logarithm add also the given logarithm of the said next less number a, so will the last sum be the logarithm of the number b proposed.

* For the demonstration of this rule, see my Mathematical Tables, p. 109, &c.

That

That is,

$$\text{Log. of } b \text{ is log. } a + \frac{n}{s} \times \left(1 + \frac{1}{3s^2} + \frac{1}{5s^4} + \frac{1}{7s^6} + \&c.\right)$$

where n denotes the constant given decimal ·8685889638 &c.

EXAMPLES.

Ex. 1. Let it be required to find the log. of the number 2. Here the given number b is 2, and the next less number a is 1, whose log. is 0; also the sum $2 + 1 = 3 = s$, and its square $s^2 = 9$. Then the operation will be as follows:

3)	·868588964	1)	·289529654 (	·289529654
9)	·289529654	3)	32169962 (	10723321
9)	32169962	5)	3574440 (	714888
9)	3574440	7)	397160 (	56737
9)	397160	9)	44129 (	4903
9)	44129	11)	4903 (	446
9)	4903	13)	545 (	42
9)	545	15)	61 (	4
9)	61			

log. of $\frac{2}{1}$ -	·301029995
add log. 1 -	000000000
log. of 2 -	·301029995

Ex. 2. To compute the logarithm of the number 3.

Here $b = 3$, the next less number $a = 2$, and the sum $a + b = 5 = s$, whose square s^2 is 25, to divide by which, always multiply by ·04. Then the operation is as follows:

5)	·868588964	1)	·173717793 (	·173717793
25)	·173717793	3)	6948712 (	2316237
25)	6948712	5)	277948 (	55590
25)	277948	7)	11118 (	1588
25)	11118	9)	445 (	50
25)	445	11)	18 (	2
	18			

log. of $\frac{3}{2}$ -	·176091260
log. of 2 add	·301029995
log. of 3 sought	·477121255

Then, because the sum of the logarithms of numbers, gives the logarithm of their product; and the difference of the logarithms, gives the logarithm of the quotient of the numbers;

numbers; from the above two logarithms, and the logarithm of 10, which is 1, we may raise a great many logarithms, as in the following examples:

EXAMPLE 3.

Because $2 \times 2 = 4$, therefore

to log. 2 -	·301029995$\frac{2}{3}$
add log. 2 -	·301029995$\frac{2}{3}$
sum is log. 4	·602059991$\frac{1}{3}$

EXAMPLE 4.

Because $2 \times 3 = 6$, therefore

to log. 2 -	·301029995
add log. 3.	·477121255
sum is log. 6.	·778151250

EXAMPLE 5.

Because $2^3 = 8$, therefore

log. 2 -	·301029995$\frac{2}{3}$
mult. by 3	3
gives log. 8	·903089987

EXAMPLE 6.

Because $3^2 = 9$, therefore

log. 3 -	·477121254$\frac{7}{10}$
mult. by 2	2
gives log. 9	·954242509

EXAMPLE 7.

Because $\frac{10}{2} = 5$, therefore

from log. 10	1·000000000
take log. 2	·301029995$\frac{2}{3}$
leaves log. 5	·698970004$\frac{1}{3}$

EXAMPLE 8.

Because $3 \times 4 = 12$, therefore

to log. 3 -	·477121255
add log. 4	·602059991
gives log. 12	1·079181246

And thus, computing, by this general rule, the logarithms to the other prime numbers, 7, 11, 13, 17, 19, 23, &c, and then using composition and division, we may easily find as many logarithms as we please, or may speedily examine any logarithm in the table *.

* There are, besides these, many other ingenious methods, which later writers have discovered for finding the logarithms of numbers, in a much easier way than by the original inventor; but, as they cannot be understood without a knowledge of some of the higher branches of the mathematics, it is thought proper to omit them, and to refer the reader to those works which are written expressly on the subject. It would likewise much exceed the limits of this compendium, to point out all the peculiar artifices that are made use of for constructing an entire table of these numbers; but any information of this kind, which the learner may wish to obtain, may be found in my Tables, before mentioned.

Description

Description and Use of the TABLE *of* LOGARITHMS.

HAVING explained the manner of forming a table of the logarithms of numbers, greater than unity; the next thing to be done is, to show how the logarithms of fractional quantities may be found. In order to this, it may be observed, that as in the former case a geometric series is supposed to increase towards the left, from unity, so in the latter case it is supposed to decrease towards the right hand, still beginning with unit; as exhibited in the general description, page 148, where the indices being made negative, still show the logarithms to which they belong. Whence it appears, that as + 1 is the log. of 10, so − 1 is the log. of $\frac{1}{10}$ or ·1; and as + 2 is the log. of 100, so − 2 is the log. of $\frac{1}{100}$ or ·01 : and so on.

Hence it appears in general, that all numbers which consist of the same figures, whether they be integral, or fractional, or mixed, will have the decimal parts of their logarithms the same, but differing only in the index, which will be more or less, and positive or negative, according to the place of the first figure of the number.

Thus, the logarithm of 2651 being 3·423410, the log. of $\frac{1}{10}$, or $\frac{1}{100}$, or $\frac{1}{1000}$, &c, part of it; will be as follows:

Numbers.	Logarithms.
2 6 5 1	3 ·4 2 3 4 1 0
2 6 5 ·1	2 ·4 2 3 4 1 0
2 6 ·5 1	1 ·4 2 3 4 1 0
2 ·6 5 1	0 ·4 2 3 4 1 0
·2 6 5 1	−1 ·4 2 3 4 1 0
·0 2 6 5 1	−2 ·4 2 3 4 1 0
·0 0 2 6 5 1	−3 ·4 2 3 4 1 0

Hence it also appears, that the index of any logarithm, is always less by 1 than the number of integer figures which the natural number consists of; or it is equal to the distance of the first figure from the place of units, or first place of integers, whether on the left, or on the right, of it: and this index is constantly to be placed on the left-hand side of the decimal part of the logarithm.

When there are integers in the given number, the index is always affirmative; but when there are no integers, the index is negative, and is to be marked by a short line drawn before it, or else above it. Thus,

A number having 1, 2, 3, 4, 5, &c, integer places,
the index of its log. is 0, 1, 2, 3, 4, &c. or 1 less than those places.

And

And a decimal fraction having its first figure in the 1st, 2d, 3d, 4th, &c, place of the decimals, has always -1, -2, -3, -4, &c, for the index of its logarithm.

It may also be observed, that though the indices of fractional quantities are negative, yet the decimal parts of their logarithms are always affirmative. And the negative mark ($-$) may be set either before the index or over it.

1. TO FIND, IN THE TABLE, THE LOGARITHM TO ANY NUMBER*.

1. *If the given Number be less than* 100, or consist of only two figures; its log. is immediately found by inspection in the first page of the table, which contains all numbers from 1 to 100, with their logs. and the index immediately annexed in the next column.

So the log. of 5 is 0·698970. The log. of 23 is 1·361728. The log. of 50 is 1·698970. And so on.

2. *If the Number be more than* 100 *but less than* 10000; that is, consisting of either three or four figures; the decimal part of the logarithm is found by inspection in the other pages of the table, standing against the given number, in this manner; viz. the first three figures of the given number in the first column of the page, and the fourth figure one of those along the top line of it; then in the angle of meeting are the last four figures of the logarithm, and the first two figures of the same at the beginning of the same line in the second column of the page: to which is to be prefixed the proper index, which is always 1 less than the number of integer figures.

So the logarithm of 251 is 2·399674, that is, the decimal ·399674 found in the table, with the index 2 prefixed, because the given number contains three integers. And the log. of 34·09 is 1·532627, that is, the decimal ·532627 found in the table, with the index 1 prefixed, because the given number contains two integers.

3. *But if the given Number contain more than four figures;* take out the logarithm of the first four figures by inspection in the table, as before, as also the next greater logarithm, subtracting the one logarithm from the other, as also their corresponding numbers the one from the other. Then say,

As the difference between the two numbers,
Is to the difference of their logarithms,
So is the remaining part of the given number,
To the proportional part of the logarithm.

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* See the table of Logarithms, after the Geometry, at the end of this volume.

Which part being added to the less logarithm, before taken out, gives the whole logarithm sought very nearly.

EXAMPLE.

To find the logarithm of the number			34·0926.
The log. of 340900, as before, is			·532627.
And log. of 341000	-		is 532754.
The diffs. are	100	and	127

Then, as 100 : 127 :: 26 : 33, the proportional part.
This added to - - - 532627, the first log.
Gives, with the index, 1·532660 for the log. of 34·0926.

4. If the number consist both of integers and fractions, or is entirely fractional; find the decimal part of the logarithm the same as if all its figures were integral; then this, having prefixed to it the proper index, will give the logarithm required.

5. And if the given number be a proper vulgar fraction: subtract the logarithm of the denominator from the logarithm of the numerator, and the remainder will be the logarithm sought; which, being that of a decimal fraction, must always have a negative index.

6. But if it be a mixed number; reduce it to an improper fraction, and find the difference of the logarithms of the numerator and denominator, in the same manner as before.

EXAMPLES.

1. To find the log. of $\frac{37}{94}$.

Log. of 37	-	1·568202
Log. of 94	-	1·973128
Dif. log. of $\frac{37}{94}$		−1·595074

Where the index 1 is negative.

2. To find the log. of $17\frac{14}{23}$.

First, $17\frac{14}{23} = \frac{405}{23}$. Then,

Log. of 405	-	2·607455
Log. of 23	-	1·361728
Dif. log. of $17\frac{14}{23}$		1·245727

II. TO FIND THE NATURAL NUMBER TO ANY GIVEN LOGARITHM.

This is to be found in the tables by the reverse method to the former, namely, by searching for the proposed logarithm among those in the table, and taking out the corresponding number by inspection, in which the proper number of integers are to be pointed off, viz. 1 more than the index. For, in finding the number answering to any given logarithm, *the index* always shows how far the first figure

must

must be removed from the place of units, viz. to the left hand, or integers, when the index is affirmative; but to the right hand, or decimals, when it is negative.

EXAMPLES.

So, the number to the log. 1·532882 is 34·11.

And the number of the log. $\bar{1}$·532882 is ·3411.

But if the logarithm cannot be exactly found in the table; take out the next greater and the next less, subtracting the one of these logarithms from the other, as also their natural numbers the one from the other, and the less logarithm from the logarithm proposed. Then say,

As the difference of the first or tabular logarithms,
Is to the difference of their natural numbers,
So is the differ. of the given log. and the least tabular log.
To their corresponding numeral difference.

Which being annexed to the least natural number above taken, gives the natural number sought, corresponding to the proposed logarithm.

EXAMPLE.

So, to find the natural number answering to the given logarithm 1·532708.

Here the next greater and next less tabular logarithms, with their corresponding numbers, are as below:

Next greater	532754	its num. 341000;	given log.		532708
Next less	532627	its num. 340900;	next less		532627
Differences	127	—	100	—	81

Then, as 127 : 100 :: 81 : 64 nearly, the numeral differ.

Therefore 34·0964 is the number sought, marking off two integers, because the index of the given logarithm is 1.

Had the index been negative, thus $\bar{1}$·532708, its corresponding number would have been ·340964, wholly decimal.

MULTIPLI-

MULTIPLICATION BY LOGARITHMS.

RULE.

TAKE out the logarithms of the factors from the table, then add them together, and their sum will be the logarithm of the product required. Then, by means of the table, take out the natural number, answering to the sum, for the product sought.

Observing to add what is to be carried from the decimal part of the logarithm to the affirmative index or indices, or else subtract it from the negative.

Also, adding the indices together when they are of the same kind, both affirmative or both negative; but subtracting the less from the greater, when the one is affirmative and the other negative, and prefixing the sign of the greater to the remainder.

EXAMPLES.

1. To multiply 23·14 by 5·062.

Numbers.	Logs.
23.14 -	1·364363
5·062 -	0·704322
Product 117·1347	2·068685

2. To multiply 2·581926 by 3·457291.

Numbers.	Logs.
2·581926 -	0·411944
3·457291 -	0·538736
Prod. 8·92648 -	0·950680

3. To mult. 3·902 and 597·16 and ·0314728 all together.

Numbers.	Logs.
3·902 -	0·591287
597·16 -	2·776091
·0314728 —	2·497935
Prod. 73·3333 -	1·865313

Here the —2 cancels the 2, and the 1 to carry from the decimals is set down.

4. To mult. 3·586, and 2·1046, and 0·8372, and 0·0294 all together.

Numbers.	Logs.
3·586 -	0·554610
2·1046 -	0·323170
0·8372	—1·922829
0·0294	—2·468347
Prod. 0·1857618	—1·268956

Here the 2 to carry cancels the —2, and there remains the —1 to set down.

DIVISION

DIVISION BY LOGARITHMS.

RULE.

FROM the logarithm of the dividend subtract the logarithm of the divisor, and the number answering to the remainder will be the quotient required.

Observing to change the sign of the index of the divisor, from affirmative to negative, or from negative to affirmative; then take the sum of the indices if they be of the same name, or their difference when of different signs, with the sign of the greater, for the index to the logarithm of the quotient.

And also, when 1 is borrowed, in the left-hand place of the decimal part of the logarithm, add it to the index of the divisor when that index is affirmative, but subtract it when negative; then let the sign of the index arising from hence be changed, and worked with as before.

EXAMPLES.

1. To divide 24163 by 4567.

	Numbers.	Logs.
Dividend	24163 -	4·383151
Divisor	4567 -	3·659631
Quot.	5·29078	0·723520

2. To divide 37·149 by 523·76.

	Numbers.	Logs.
Dividend	37·149 -	1·569947
Divisor	523·76 -	2·719132
Quot.	·0709275	−2·850815

3. Divide ·06314 by ·007241.

	Numbers.	Logs.
Divid.	·06314	−2·800305
Divisor	·007241	−3·859799
Quot.	8·71979	0·940506

Here 1 carried from the decimals to the −3, makes it become −2, which taken from the other −2; leaves 0 remaining.

4. To divide ·7438 by 12·9476.

	Numbers.	Logs.
Divid.	·7438	−1·871456
Divisor	12·9476	1·112189
Quot.	·057447	−2·759267

Here the 1 taken from the −1, makes it become −2, to set down.

Note. As to the Rule-of-Three, or Rule of Proportion, it is performed by adding the logarithms of the 2d and 3d terms, and subtracting that of the first term from their sum.

INVOLUTION BY LOGARITHMS.

RULE.

Take out the logarithm of the given number from the table. Multiply the log. thus found, by the index of the power proposed. Find the number answering to the product, and it will be the power required.

Note. In multiplying a logarithm with a negative index, by an affirmative number, the product will be negative. But what is to be carried from the decimal part of the logarithm, will always be affirmative. And therefore their difference will be the index of the product, and is always to be made of the same kind with the greater.

EXAMPLES.

1. To square the number 2·5791.

	Numb.	Log.
Root	2·5791 - -	0·411468
	The index - -	2
Power	6·65174	0·822936

2. To find the cube of 3·07146.

	Numb.	Log.
Root	3·07146 - -	0·487345
	The index - -	3
Power	28·9758	1·462035

3. To raise ·09163 to the 4th power.

	Numb.	Log.
Root	·09163	—2·962038
	The index - -	4
Pow.	·000070494	—5·848152

Here 4 times the negative index being —8, and 3 to carry, the difference —5 is the index of the product.

4. To raise 1·0045 to the 365th power.

	Numb.	Log.
Root	1·0045 - -	0·001950
	The index - -	365
		9750
		11700
		5850
Power	5·14932	0·711750

EVOLUTION BY LOGARITHMS.

TAKE the log: of the given number out of the table. Divide the log. thus found by the index of the root. Then the number answering to the quotient, will be the root.

Note. When the index of the logarithm, to be divided, is negative, and does not exactly contain the divisor, without some remainder, increase the index by such a number as will make it exactly divisible by the index, carrying the units borrowed, as so many tens, to the left-hand place of the decimal, and then divide as in whole numbers.

Ex. 1. To find the square root of 365.

	Numb.	Log.
Power	365	2)2·562293
Root	19·10496	1·281146$\frac{1}{2}$

Ex. 2. To find the 3d root of 12345.

	Numb.	Log.
Power	12345	3) 4·091491
Root	23·1116	1·363830$\frac{1}{3}$

Ex. 3. To find the 10th root of 2.

	Numb.	Log.
Power	2, -	10) 0·301030
Root	1·071773	0·030103

Ex. 4. To find the 365th root of 1·045.

	Numb.	Log.
Power	1·045	365)0·019116
Root	1·000121	0·0000052$\frac{1}{2}$

Ex. 5. To find $\sqrt{\cdot 093}$.

	Numb.	Log.
Power	·093	2)−2·968483
Root	·304959	−1·484241$\frac{1}{2}$

Here the divisor 2 is contained exactly once in the negative index −2, and therefore the index of the quotient is −1.

Ex. 6. To find the $\sqrt[3]{\cdot 00048}$.

	Numb.	Log.
Power	·00048	3)−4·681241
Root	·0782973	−2·893747

Here the divisor 3, not being exactly contained in −4, it is augmented by 2, to make up 6, in which the divisor is contained just 2 times; then the 2, thus borrowed, being carried to the decimal figure 6, makes 26, which divided by 3, gives 8, &c.

Ex. 7. To find $3·1416 \times 82 \times \frac{11}{44}$.

Ex. 8. To find $·02916 \times 751·3 \times \frac{5}{941}$.

Ex. 9. As 7241 : 3·58 :: 20·46 : ?

Ex. 10. As $\sqrt{724}$: $\sqrt{\frac{53}{13}}$ :: 6·927 : ?

ALGEBRA.

ALGEBRA.

DEFINITIONS AND NOTATION.

1. ALGEBRA is the science of computing by symbols. It is sometimes also called Analysis; and is a general kind of arithmetic, or universal way of computation.

2. In this science, quantities of all kinds are represented by the letters of the alphabet. And the operations to be performed with them, as addition or subtraction, &c, are denoted by certain simple characters, instead of being expressed by words at length.

3. In algebraical questions, some quantities are known or given, viz. those whose values are known: and others unknown, or are to be found out, viz. those whose values are not known. The former of these are represented by the leading letters of the alphabet, a, b, c, d, &c; and the latter, or unknown quantities, by the final letters, z, y, x, u, &c.

4. The characters used to denote the operations, are chiefly the following:

$+$ signifies addition, and is named *plus*.
$-$ signifies subtraction, and is named *minus*.
$\times$ or . signifies multiplication, and is named *into*.
$\div$ signifies division, and is named *by*.
$\surd$ signifies the square root; $\sqrt[3]{}$ the cube root; $\sqrt[4]{}$ the 4th root, &c; and $\sqrt[n]{}$ the nth root.
: :: : signifies proportion.
$=$ signifies equality, and is named *equal to*.
And so on for other operations.

Thus $a + b$ denotes that the number represented by b is to be added to that represented by a.

$a - b$ denotes, that the number represented by b is to be subtracted from that represented by a.

$a \backsim b$ denotes the difference of a and b, when it is not known which is the greater.

ab, or $a \times b$, or $a.b$, expresses the product, by multiplication, of the numbers represented by a and b.

$a \div b$, or $\frac{a}{b}$, denotes, that the number represented by a is to be divided by that which is expressed by b.

$a : b :: c : d$, signifies that a is in the same proportion to b, as c is to d.

$x = a - b + c$ is an equation, expressing that x is equal to the difference of a and b, added to the quantity c.

$\sqrt{a}$, or $a^{\frac{1}{2}}$, denotes the square root of a; $\sqrt[3]{a}$, or $a^{\frac{1}{3}}$, the cube root of a; and $\sqrt[3]{a^2}$ or $a^{\frac{2}{3}}$ the cube root of the square of a; also $\sqrt[m]{a}$, or $a^{\frac{1}{m}}$, is the mth root of a; and $\sqrt[m]{a^n}$ or $a^{\frac{n}{m}}$ is the nth power of the mth root of a, or it is a to the $\frac{n}{m}$ power.

a^2 denotes the square of a; a^3 the cube of a; a^4 the fourth power of a; and a^n the nth power of a.

$\overline{a+b} \times c$, or $(a+b)\,c$, denotes the product of the compound quantity $a+b$ multiplied by the simple quantity c. Using the bar ——, or the parenthesis ( ) as a vinculum, to connect several simple quantities into one compound.

$\overline{a+b} \div \overline{a-b}$, or $\frac{a+b}{a-b}$, expressed like a fraction, means the quotient of $a+b$ divided by $a-b$.

$\sqrt{ab+cd}$, or $(ab+cd)^{\frac{1}{2}}$, is the square root of the compound quantity $ab+cd$. And $c\sqrt{ab+cd}$, or $c\,(ab+cd)^{\frac{1}{2}}$, denotes the product of c into the square root of the compound quantity $ab+cd$.

$\overline{a+b-c}^{\,3}$, or $(a+b-c)^3$, denotes the cube, or third power, of the compound quantity $a+b-c$.

$3a$ denotes that the quantity a is to be taken 3 times, and $4\,(a+b)$ is 4 times $a+b$. And these numbers, 3 or 4, showing how often the quantities are to be taken, or multiplied, are called Co-efficients.

Also $\frac{3}{4}x$ denotes that x is multipled by $\frac{3}{4}$; thus $\frac{3}{4} \times x$ or $\frac{3}{4}x$.

5. Like Quantities, are those which consist of the same letters, and powers. As a and $3a$; or $2ab$ and $4ab$; or $3a^2bc$ and $-5a^2bc$.

6. Unlike Quantities, are those which consist of different letters, or different powers. As a and b; or $2a$ and a^2; or $3ab^2$ and $3abc$.

7. Simple

7. Simple Quantities, are those which consist of one term only. As $3a$, or $5ab$, or $6abc^2$.

8. Compound Quantities are those which consist of two or more terms. As $a + b$, or $2a - 3c$, or $a + 2b - 3c$.

9. And when the compound quantity consists of two terms, it is called a Binomial, as $a + b$; when of three terms, it is a Trinomial, as $a + 2b - 3c$; when of four terms, a Quadrinomial, as $2a - 3b + c - 4d$; and so on. Also, a Multinomial or Polynomial, consists of many terms.

10. A Residual Quantity, is a binomial having one of the terms negative. As $a - 2b$.

11. Positive or Affirmative Quantities, are those which are to be added, or have the sign $+$. As a or $+ a$, or ab: for when a quantity is found without a sign, it is understood to be positive, or have the sign $+$ prefixed.

12. Negative Quantities, are those which are to be subtracted. As $- a$, or $-2ab$, or $-3ab^2$.

13. Like Signs, are either all positive ($+$), or all negative ($-$).

14. Unlike Signs, are when some are positive ($+$), and others negative ($-$).

15. The Co-efficient of any quantity, as shown above, is the number prefixed to it. As 3, in the quantity $3ab$.

16. The Power of a quantity (a), is its square (a^2), or cube (a^3), or biquadrate (a^4), &c; called also, the 2d power, or 3d power, or 4th power, &c.

17. The Index or Exponent, is the number which denotes the power or root of a quantity. So 2 is the exponent of the square or second power a^2; and 3 is the index of the cube or 3d power; and $\frac{1}{2}$ is the index of the square root, $a^{\frac{1}{2}}$ or $\sqrt{a}$; and $\frac{1}{3}$ is the index of the cube root, $a^{\frac{1}{3}}$, or $\sqrt[3]{a}$.

18. A Rational Quantity, is that which has no radical sign ($\sqrt{\ }$) or index annexed to it. As a, or $3ab$.

19. An Irrational Quantity, or Surd, is that which has not an exact root, or is expressed by means of the radical sign $\sqrt{\ }$. As $\sqrt{2}$, or $\sqrt{a}$, or $\sqrt[3]{a^2}$, or $ab^{\frac{1}{2}}$.

20. The Reciprocal of any quantity, is that quantity inverted, or unity divided by it. So, the reciprocal of a, or $\frac{a}{1}$, is $\frac{1}{a}$, and the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$.

 21. The

21. The letters by which any simple quantity is expressed, may be ranged according to any order at pleasure. So the product of a and b, may be either expressed by ab, or ba; and the product of a, b, and c, by either abc, or acb, or bac, or bca, or cab, or cba; as it matters not which quantities are placed or multiplied first. But it will be sometimes found convenient in long operations, to place the several letters according to their order in the alphabet, as abc, which order also occurs most easily or naturally to the mind.

22. Likewise, the several members, or terms, of which a compound quantity is composed, may be disposed in any order at pleasure, without altering the value of the signification of the whole. Thus, $3a - 2ab + 4abc$ may also be written $3a + 4abc - 2ab$, or $4abc + 3a - 2ab$, or $-2ab + 3a + 4abc$, &c; for all these represent the same thing, namely, the quantity which remains, when the quantity or term $2ab$ is subtracted from the sum of the terms or quantities $3a$ and $4abc$. But it is most usual and natural, to begin with a positive term, and with the first letters of the alphabet.

SOME EXAMPLES FOR PRACTICE,

In finding the numeral values of various expressions, or combinations, of quantities.

Supposing $a = 6$, and $b = 5$, and $c = 4$, and $d = 1$, and $e = 0$. Then

1. Will $a^2 + 3ab - c^2 = 36 + 90 - 16 = 110$.

2. And $2a^3 - 3a^2b + c^3 = 432 - 540 + 64 = -44$.

3. And $a^2 \times \overline{a + b} - 2abc = 36 \times 11 - 240 = 156$.

4. And $\dfrac{a^3}{a + 3c} + c^2 = \dfrac{216}{18} + 16 = 12 + 16 = 28$.

5. And $\sqrt{2ac + c^2}$ or $\overline{2ac + c^2}|^{\frac{1}{2}} = \sqrt{64} = 8$.

6. And $\sqrt{c} + \dfrac{2bc}{\sqrt{2ac + c^2}} = 2 + \dfrac{40}{8} = 7$.

7. And $\dfrac{a^2 - \sqrt{b^2 - ac}}{2a - \sqrt{b^2 + ac}} = \dfrac{36 - 1}{12 - 7} = \dfrac{35}{5} = 7$.

8. And $\sqrt{b^2 - ac} + \sqrt{2ac + c^2} = 1 + 8 = 9$.

9. And $\sqrt{\overline{b^2 - ac} + \sqrt{2ac + c^2}} = \sqrt{25 - 24 + 8} = 3$.

10. And $a^2b + c - d = 183$.

11. And $9ab - 10b^2 + c = 24$.

12. And

12. And $\frac{a^2b}{c} \times d = 45$.

13. And $\frac{a+b}{c} \times \frac{b}{d} = 13\frac{3}{4}$.

14. And $\frac{a+b}{c} - \frac{a-b}{d} = 1\frac{3}{4}$.

15. And $\frac{a^2b}{c} + e = 45$.

16. And $\frac{a^2b}{c} \times e = 0$.

17. And $\overline{b-c} \times \overline{d-e} = 1$.

18. And $\overline{a+b-c-d} = 8$.

19. And $\overline{a+b} - c - d = 6$.

20. And $a^2c \times d^3 = 144$.

21. And $acd - d = 23$.

22. And $a^2e + b^2e + d = 1$.

23. And $\frac{b-e}{d-e} \times \frac{a+b}{c-d} = 18\frac{1}{3}$.

24. And $\sqrt{a^2+b^2} - \sqrt{a^2-b^2} = 4{\cdot}4936249$.

25. And $3ac^2 + \sqrt[3]{a^3-b^3} = 292{\cdot}497942$.

26. And $4a^2 - 3a\sqrt{a^2 - \frac{2}{3}ab} = 72$.

ADDITION.

Addition, in Algebra, is the connecting the quantities together by their proper signs, and incorporating or uniting into one term or sum, such as are similar, and can be united. As $3a + 2b - 2a = a + 2b$, the sum.

The rule of addition in algebra, may be divided into three cases: one, when the quantities are like, and their signs like also; a second, when the quantities are like, but their signs unlike; and the third, when the quantities are unlike. Which are performed as follows*.

CASE

* The reasons on which these operations are founded, will readily appear, by a little reflection on the nature of the quantities to be

CASE I.

When the Quantities are Like, and have Like Signs.

ADD the co-efficients together, and set down the sum; after which set the common letter or letters of the like quantities, and prefix the common sign + or −.

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be added, or collected together. For, with regard to the first example, where the quantities are $3a$ and $5a$, whatever a represents in the one term, it will represent the same thing in the other; so that 3 times any thing and 5 times the same thing, collected together, must needs make 8 times that thing. As if a denote a shilling; then $3a$ is 3 shillings, and $5a$ is 5 shillings, and their sum 8 shillings. In like manner, $-2ab$ and $-7ab$, or -2 times any thing, and -7 times the same thing, make -9 times that thing.

As to the second case, in which the quantities are like, but the signs unlike; the reason of its operation will easily appear, by reflecting, that addition means only the uniting of quantities together by means of the arithmetical operations denoted by their signs + and −, or of addition and subtraction; which being of contrary or opposite natures, the one co-efficient must be subtracted from the other, to obtain the incorporated or united mass.

As to the third case, where the quantities are unlike, it is plain that such quantities cannot be united into one, or otherwise added, than by means of their signs: thus, for example, if a be supposed to represent a crown, and b a shilling; then the sum of a and b can be neither $2a$ nor $2b$, that is neither 2 crowns nor 2 shillings, but only 1 crown plus 1 shilling, that is $a + b$.

In this rule, the word *addition* is not very properly used; being much too limited to express the operation here performed. The business of this operation is to incorporate into one mass, or algebraic expression, different algebraic quantities, as far as an actual incorporation or union is possible; and to retain the algebraic marks for doing it, in cases where the former is not possible. When we have several quantities, some affirmative and some negative; and the relation of these quantities can in the whole or in part be discovered; such incorporation of two or more quantities into one, is plainly effected by the foregoing rules.

It may seem a paradox, that what is called addition in algebra, should sometimes mean addition, and sometimes subtraction. But the paradox wholly arises from the scantiness of the name given to the algebraic process; from employing an old term in a new and more enlarged sense. Instead of addition, call it incorporation, or union, or striking a balance, or any name to which a more extensive idea may be annexed, than that which is usually implied by the word addition; and the paradox vanishes.

Thus,

Thus, $3a$ added to $5a$, makes $8a$.
And $-2ab$ added to $-7ab$, makes $-9ab$.
And $5a + 7b$ added to $7a + 3b$, makes $12a + 10b$

OTHER EXAMPLES FOR PRACTICE.

$3a$	$-3bx$	bxy
$9a$	$-5bx$	$2bxy$
$5a$	$-4bx$	$5bxy$
$12a$	$-2bx$	bxy
a	$-7bx$	$3bxy$
$2a$	$-bx$	$6bxy$
$31a$	$-22bx$	$17bxy$

$3z$	$3x^2 + 5xy$	$2ax - 4y$
$2z$	$x^2 + xy$	$4ax - y$
$4z$	$2x^2 + 4xy$	$ax - 3y$
z	$5x^2 + 2xy$	$5ax - 5y$
$5z$	$4x^2 + 3xy$	$7ax - 2y$
$15z$	$15x^2 + 15xy$	$19ax - 15y$

$5xy$	$-12y^2$	$4a - 4b$
$14xy$	$-7y^2$	$5a - 5b$
$22xy$	$-2y^2$	$6a - b$
$17xy$	$-4y^2$	$8a - 2b$
$1\frac{1}{2}xy$	$-y^2$	$2a - 7b$
$\frac{1}{2}xy$	$-3y^2$	$8a - b$

$30 - 13x^{\frac{1}{2}} - 3xy$	$5xy - 3x + 4ab$
$23 - 10x^{\frac{1}{2}} - 4xy$	$8xy - 4x + 3ab$
$14 - 14x^{\frac{1}{2}} - 7xy$	$3xy - 5x + 5ab$
$10 - 16x^{\frac{1}{2}} - 5xy$	$xy - 2x + ab$
$16 - 20x^{\frac{1}{2}} - xy$	$4xy - x + 7ab$

CASE II.

When the Quantities are Like, but have Unlike Signs:

Add all the affirmative co-efficients into one sum, and all the negative ones into another, when there are several of a kind. Then subtract the less sum, or the less co-efficient, from the greater; and to the remainder prefix the sign of the greater, and subjoin the common quantity or letters.

So $+ 5a$ and $- 3a$, united, make $+ 2a$.
And $- 5a$ and $+ 3a$, united, make $- 2a$.

OTHER EXAMPLES FOR PRACTICE.

$-5a$	$+3ax^2$	$+8x^3+3y$
$+4a$	$+4ax^2$	$-5x^3+4y$
$+6a$	$-8ax^2$	$-16x^3+5y$
$-3a$	$-6ax^2$	$+3x^3-7y$
$+a$	$+5ax^2$	$+2x^3-2y$
$+3a$	$+2ax^2$	$-8x^3+10y$

$-3a^2$	$+3b^2y^3$	$+4ab+4$
$-5a^2$	$+9b^2y^3$	$-4ab+12$
$-10a^2$	$-10b^2y^3$	$+7ab-14$
$+10a^2$	$-19b^2y^3$	$+ab+3$
$+14a^2$	$-2b^2y^3$	$-5ab-10$

$-3ax^{\frac{1}{2}}$	$+10\sqrt{ax}$	$+3y+4ax^{\frac{1}{2}}$
$+ax^{\frac{1}{2}}$	$-3\sqrt{ax}$	$-y-5ax^{\frac{1}{2}}$
$+5ax^{\frac{1}{2}}$	$+4\sqrt{ax}$	$+4y+2ax^{\frac{1}{2}}$
$-6ax^{\frac{1}{2}}$	$-12\sqrt{ax}$	$-2y+6ax^{\frac{1}{2}}$

CASE III.

When the Quantities are Unlike.

HAVING collected together all the like quantities, as in the two foregoing cases, set down those that are unlike, one after another, with their proper signs.

EXAMPLES.

$$\begin{array}{r}3xy\\2ax\\-5xy\\6ax\\\hline -2xy+8ax\\\hline\end{array}\qquad\begin{array}{r}6xy-12x^2\\-4x^2+3xy\\+4x^2-2xy\\-3xy+4x^2\\\hline 4xy-8x^2\\\hline\end{array}\qquad\begin{array}{l}4ax-130+3x^{\frac{1}{2}}\\5x^2+3ax+9x^2\\7xy-4x^{\frac{1}{2}}+90\\\sqrt{x}+40-6x^2\\\hline 7ax+8x^2+7xy\\\hline\end{array}$$

$$\begin{array}{r}9x^2y^2\\-7x^2y\\+3axy\\-4x^2y\\\hline\\\hline\end{array}\qquad\begin{array}{l}14ax-2x^2\\5ax+3xy\\8y^2-4ax\\3x^2+26\\\hline\\\hline\end{array}\qquad\begin{array}{l}9+10\sqrt{ax}-5y\\2x+7\sqrt{xy}+5y\\5y+3\sqrt{ax}-4y\\10-4\sqrt{ax}+4y\\\hline\\\hline\end{array}$$

$$\begin{array}{r}4x^2y\\-6xy^2\\+3y^2x\\-7x^2y\\\hline\\\hline\end{array}\qquad\begin{array}{l}4\sqrt{x}-3y\\2\sqrt{xy}+14x\\3x+2y\\-9+3\sqrt{xy}\\\hline\\\hline\end{array}\qquad\begin{array}{l}3a^2+9+x^{\frac{1}{2}}-4\\2a-8+2a^2-3x\\4x^2-2a^2+18-7\\-12+a-3x^2-2y\\\hline\\\hline\end{array}$$

Add $a+b$ and $3a-5b$ together.
Add $5a-8x$ and $3a-4x$ together.
Add $6x-5b+a+8$ to $-5a-4x+4b-3$.
Add $a+2b-3c-10$ to $3b-4a+5c+10$ and $5b-c$.
Add $a+b$ and $a-b$ together.
Add $3a+b-10$ to $c-d-a$ and $-4c+2a-3b-7$.
Add $3a^2+b^2-c$ to $2ab-3a^2+bc-b$.
Add $a^3+b^2c-b^2$ to $ab^2-abc+b^2$.
Add $9a-8b+10x-6d-7c+50$ to $2x-3a-5c+4b+6d-10$.

SUBTRACTION.

Set down in one line the first quantities from which the subtraction is to be made; and underneath them place all the other quantities composing the subtrahend: ranging the like quantities under each other, as in Addition.

Then change all the signs (+ and −) of the lower line, or conceive them to be changed; after which, collect all the terms together as in the cases of Addition *.

EXAMPLES.

From	$7a^2-3b$	$9x^2-4y+8$	$8xy-3+6x-y$
Take	$3a^2-8b$	$6x^2+5y-4$	$4xy-7-6x-4y$
Rem.	$4a^2+5b$	$3x^2-9y+12$	$4xy+4+12x+3y$
From	$5xy-6$	$4y^2-3y-4$	$-20-6x-5xy$
Take	$-2xy+6$	$2y^2+2y+4$	$3xy-9x+8-2ay$
Rem.	$7xy-12$	$2y^2-5y-8$	$-28+3x-8xy+2ay$
From	$8x^2y+6$	$5\sqrt{xy}+2x\sqrt{xy}$	$7x^2+2\sqrt{x}-18+3b$
Take	$-2x^2y+2$	$7\sqrt{xy}+3-2xy$	$9x^2-12+5b+x^{\frac{1}{2}}$
Rem.			

* This rule is founded on the consideration, that addition and subtraction are opposite to each other in their nature and operation, as are the signs + and −, by which they are expressed and represented. So that, since to unite a negative quantity with a positive one of the same kind, has the effect of diminishing it, or subducting an equal positive one from it, therefore to subtract a positive (which is the opposite of uniting or adding) is to add the equal negative quantity. In like manner, to subtract a negative quantity, is the same in effect as to add or unite an equal positive one. So that, by changing the sign of a quantity from + to −, or from − to +, changes its nature from a subductive quantity to an additive one; and any quantity is in effect subtracted, by barely changing its sign.

$5xy - 30$	$7x^3 - 2(a+b)$	$3xy^3 + 20a\sqrt{(xy+10)}$
$7xy - 50$	$2x^2 - 4(a+b)$	$4x^2y^2 + 12a\sqrt{(xy+10)}$

From $a+b$, take $a-b$.
From $4a+4b$, take $b+a$.
From $4a-4b$, take $3a+5b$.
From $8a-12x$, take $4a-3x$.
From $2x-4a-2b+5$, take $8-5b+a+6x$.
From $3a+b+c-d-10$, take $c+2a-d$.
From $3a+b+c-d-10$, take $b-10+3a$.
From $2ab+b^2-4c+bc-b$, take $3a^2-c+b^2$.
From $a^3+3b^2c+ab^2-abc$, take b^3+ab^2-abc.
From $12x+6a-4b+40$, take $4b-3a+4x+6d-10$.
From $2x-3a+4b+6c-50$, take $9a+x+6b-6c-40$.
From $6a-4b-12c+12x$, take $2x-8a+4b-5c$.

MULTIPLICATION.

This consists of several cases, according as the factors are simple or compound quantities.

CASE I. *When both the Factors are Simple Quantities:*

FIRST multiply the co-efficients of the two terms together, then to the product annex all the letters in those terms, which will give the whole product required.

Note *. Like signs, in the factors, produce +, and unlike signs −, in the products.

EXAMPLES.

* That this rule for the signs is true, may be thus shown.

1. When $+a$ is to be multiplied by $+c$; the meaning is, that $+a$ is to be taken as many times as there are units in c; and since the sum of any number of positive terms is positive, it follows that $+a \times +c$ makes $+ac$.

2. When

EXAMPLES.

$10a$	$-3a$	$7a$	$-6x$
$2b$	$+2b$	$-4c$	$-4a$
$20ab$	$-6ab$	$-28ac$	$+24ax$
$4ac$	$9a^2x$	$-2x^2y$	$-4xy$
$-3ab$	$4x$	$3xy^2$	$-xy$
$-12a^2bc$	$36a^2x^2$	$-6x^3y^3$	$+4x^2y^2$
$-3ax$	$-ax$	$+3xy$	$-5xyz$
$4x$	$-6c$	-4	$-5ax$

CASE II.

When one of the Factors is a Compound Quantity;

MULTIPLY every term of the multiplicand, or compound quantity, separately, by the multiplier, as in the former case; placing the products one after another, with the proper signs; and the result will be the whole product required.

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2. When two quantities are to be multiplied together, the result will be exactly the same, in whatever order they are placed; for a times c is the same as c times a, and therefore, when $-a$ is to be multiplied by $+c$, or $+c$ by $-a$: this is the same thing as taking $-a$ as many times as there are units in $+c$; and as the sum of any number of negative terms is negative, it follows that $-a \times +c$, or $+a \times -c$ make or produce $-ac$.

3. When $-a$ is to be multiplied by $-c$: here $-a$ is to be subtracted as often as there are units in c: but subtracting negatives is the same thing as adding affirmatives, by the demonstration of the rule for subtraction; consequently the product is c times a, or $+ac$.

Otherwise. Since $a - a = 0$, therefore $(a - a) \times -c$ is also $= 0$, because 0 multiplied by any quantity, is still but 0; and since the first term of the product, or $a \times -c$ is $= -ac$, by the second case; therefore the last term of the product, or $-a \times -c$, must be $+ac$, to make the sum $= 0$, or $-ac + ac = 0$; that is, $-a \times -c = +ac$.

EXAMPLES.

EXAMPLES.

$$\begin{array}{l} 5a-3c \\ 2a \\ \hline 10a^2-6ac \\ \hline \end{array} \qquad \begin{array}{l} 3ac-4b \\ 3a \\ \hline 9a^2c-12ab \\ \hline \end{array} \qquad \begin{array}{l} 2a^2-3c+5 \\ bc \\ \hline 2a^2bc-3bc^2+5bc \\ \hline \end{array}$$

$$\begin{array}{l} 12x-2ac \\ 4a \\ \hline \\ \hline \end{array} \qquad \begin{array}{l} 25c-7b \\ -2a \\ \hline \\ \hline \end{array} \qquad \begin{array}{l} 4x-b+3ab \\ 2ab \\ \hline \\ \hline \end{array}$$

$$\begin{array}{l} 3c^2+x \\ 4xy \\ \hline \\ \hline \end{array} \qquad \begin{array}{l} 10x^2-3y^2 \\ -4x^2 \\ \hline \\ \hline \end{array} \qquad \begin{array}{l} 3a^2-2x^2-6b \\ 2ax^2 \\ \hline \\ \hline \end{array}$$

CASE III.

When both the Factors are Compound Quantities;

MULTIPLY every term of the multiplier by every term of the multiplicand, separately; setting down the products one after or under another, with their proper signs; and add the several lines of products all together for the whole product required.

$$\begin{array}{l} a+b \\ a+b \\ \hline a^2+ab \\ \quad +ab+b^2 \\ \hline a^2+2ab+b^2 \\ \hline \end{array} \qquad \begin{array}{l} 3x+2y \\ 4x-5y \\ \hline 12x^2+8xy \\ \quad -15xy-10y^2 \\ \hline 12x^2-7xy-10y^2 \\ \hline \end{array} \qquad \begin{array}{l} 2x^2+xy-2y^2 \\ 3x-3y \\ \hline 6x^3+3x^2y-6xy^2 \\ \quad -6x^2y-3xy^2+6y^3 \\ \hline 6x^3-3x^2y-9xy^2+6y^3 \\ \hline \end{array}$$

$$\begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \quad -ab-b^2 \\ \hline a^2 \quad * \quad -b^2 \\ \hline \end{array} \qquad \begin{array}{l} x^2+y \\ x^2+y \\ \hline x^4+yx^2 \\ \quad +yx^2+y^2 \\ \hline x^4+2yx^2+y^2 \\ \hline \end{array} \qquad \begin{array}{l} a^2+ab+b^2 \\ a-b \\ \hline a^3+a^2b+ab^2 \\ \quad -a^2b-ab^2-b^3 \\ \hline a^3 \quad * \quad * \quad -b^3 \\ \hline \end{array}$$

Note. In the multiplication of compound quantities, it is the best way to set them down in order, according to the powers and the letters of the alphabet. And in multiplying them, begin at the left-hand side, and multiply from the left hand towards the right, in the manner that we write, which is contrary to the way of multiplying numbers. But in setting down the several products, as they arise, in the second and following lines, range them under the like terms in the lines above, when there are such like quantities; which is the easiest way for adding them up together.

In many cases, the multiplication of compound quantities is only to be performed by setting them down one after another, each within or under a vinculum, with a sign of multiplication between them. As $(a + b) \times (a - b) \times 3ab$, or $\overline{a + b} \,.\, \overline{a - b} \,.\, 3ab$.

EXAMPLES FOR PRACTICE.

1. Multiply $10ac$ by $2a$. Ans. $20a^2c$.
2. Multiply $3a^2 - 2b$ by $3b$. Ans. $9a^2b - 6b^2$.
3. Multiply $3a + 2b$ by $3a - 2b$. Ans. $9a^2 - 4b^2$.
4. Multiply $x^2 - xy + y^2$ by $x + y$. Ans. $x^3 + y^3$.
5. Multiply $a^3 + a^2b + ab^2 + b^3$ by $a - b$. Ans. $a^4 - b^4$.
6. Multiply $a^2 + ab + b^2$ by $a^2 - ab + b^2$.
7. Multiply $3x^2 - 2xy + 5$ by $x^2 + 2xy - 6$.
8. Multiply $3a^2 - 2ax + 5x^2$ by $3a^2 - 4ax - 7x^2$.
9. Multiply $3x^3 + 2x^2y^2 + 3y^3$ by $2x^3 - 3x^2y^2 + 3y^3$.
10. Multiply $a^2 + ab + b^2$ by $a - 2b$.

DIVISION.

DIVISION in Algebra, like that in numbers, is the converse of multiplication; and it is performed like that of numbers also, by beginning at the left-hand side, and dividing all the parts of the dividend by the divisor, when they can be so divided; or else by setting them down like a fraction, the dividend over the divisor, and then abbreviating the fraction as much as can be done. This will naturally divide into the following particular cases.

CASE

CASE I.

When the Divisor and Dividend are both Simple Quantities;

Set the terms both down as in division of numbers, either the divisor before the dividend, or below it, like the denominator of a fraction. Then abbreviate these terms as much as can be done, by cancelling or striking out all the letters that are common to them both, and also dividing the one co-efficient by the other, or abbreviating them after the manner of a fraction, by dividing them by their common measure.

Note. Like signs in the two factors make + in the quotient; and unlike signs make −; the same as in multiplication *.

EXAMPLES.

1. To divide $6ab$ by $3a$.

Here $6ab \div 3a$, or $3a$) $6ab$ (or $\frac{6ab}{3a} = 2b$.

2. Also $c \div c = \frac{c}{c} = 1$; and $abx \div bxy = \frac{abx}{bxy} = \frac{a}{y}$.

3. Divide $16x^2$ by $8x$. Ans. $2x$.

4. Divide $12a^2x^2$ by $-3a^2x$. Ans. $-4x$.

5. Divide $-15ay^2$ by $3ay$. Ans. $-5y$.

6. Divide $-18ax^2y$ by $-8axz$. Ans. $\frac{9xy}{4z}$.

---

* Because the divisor multiplied by the quotient, must produce the dividend. Therefore,

1. When both the terms are +, the quotient must be +; because + in the divisor × + in the quotient, produces + in the dividend.

2. When the terms are both −, the quotient is also +; because − in the divisor × + in the quotient, produces − in the dividend.

3. When one term is + and the other −, the quotient must be −; because + in the divisor × − in the quotient produces − in the dividend, or − in the divisor × + in the quotient gives − in the dividend.

So that the rule is general; viz. that like signs give +, and unlike signs give −, in the quotient.

CASE II.

When the Dividend is a Compound Quantity, and the Divisor a Simple one:

Divide every term of the dividend by the divisor, as in the former case.

EXAMPLES.

1. $(ab + b^2) \div 2b$, or $\frac{ab + b^2}{2b} = \frac{a + b}{2} = \frac{1}{2}a + \frac{1}{2}b$.

2. $(10ab + 15ax) \div 5a$, or $\frac{10ab + 15ax}{5a} = 2b + 3x$.

3. $(30az - 48z) \div z$, or $\frac{30az - 48z}{z} = 30a - 48$.

4. Divide $6ab - 8ax + a$ by $2a$.
5. Divide $3x^2 - 15 + 6x + 6a$ by $3x$.
6. Divide $6abc + 12abx - 9a^2b$ by $3ab$.
7. Divide $10a^2x - 15x^2 - 25x$ by $5x$.
8. Divide $15a^2bc - 15acx^2 + 5ad^2$ by $-5ac$.
9. Divide $15a + 3ay - 18y^2$ by $21a$.
10. Divide $-20a^2b^2 + 60ab^3$ by $-6ab$.

CASE III.

When the Divisor and Dividend are both Compound Quantities:

1. Set them down as in common division of numbers, the divisor before the dividend, with a small curved line between them, and ranging the terms according to the powers of some one of the letters in both, the higher powers before the lower.

2. Divide the first term of the dividend by the first term of the divisor, as in the first case, and set the result in the quotient.

3. Multiply the whole divisor by the term thus found, and subtract the result from the dividend.

4. To this remainder bring down as many terms of the dividend as are requisite for the next operation, dividing as before; and so on to the end, as in common arithmetic.

Note.

Note. If the divisor be not exactly contained in the dividend, the quantity which remains after the operation is finished, may be placed over the divisor, like a vulgar fraction, and set down at the end of the quotient, as in common arithmetic.

EXAMPLES.

$$
\begin{array}{rl}
a-b\,) & a^2-2ab+b^2\ (a+b \\
 & a^2-ab \\
\hline
 & \quad -ab+b^2 \\
 & \quad -ab+b^2 \\
\hline
\end{array}
$$

$$
\begin{array}{rl}
a-c\,) & a^3-4a^2c+4ac^2-c^3\ (a^2-3ac+c^2 \\
 & a^3-a^2c \\
\hline
 & \quad -3a^2c+4ac^2 \\
 & \quad -3a^2c+3ac^2 \\
\hline
 & \qquad\qquad ac^2-c^3 \\
 & \qquad\qquad ac^2-c^3 \\
\hline
\end{array}
$$

$$
\begin{array}{rl}
a-2\,) & a^3-6a^2+12a-8\ (a^2-4a+4 \\
 & a^3-2a^2 \\
\hline
 & \quad -4a^2+12a \\
 & \quad -4a^2+8a \\
\hline
 & \qquad\qquad 4a-8 \\
 & \qquad\qquad 4a-8 \\
\hline
\end{array}
$$

$$
\begin{array}{rl}
a+z\,) & a^3+z^3\ (a^2-az+z^2 \\
 & a^3+a^2z \\
\hline
 & \quad -a^2z+z^3 \\
 & \quad -a^2z-az^2 \\
\hline
 & \qquad\qquad az^2+z^3 \\
 & \qquad\qquad az^2+z^3 \\
\hline
\end{array}
$$

$$a^4 - 3x^4 \; (\; a^3 - a^2x + ax^2 - x^3 - \frac{2x^4}{a+x}$$

$$\begin{array}{l}
a^4 + a^3x \\
\hline
-\; a^3x - 3x^4 \\
-\; a^3x - a^2x^2 \\
\hline
\qquad a^2x^2 - 3x^4 \\
\qquad a^2x^2 + ax^3 \\
\hline
\qquad\quad -ax^3 - 3x^4 \\
\qquad\quad -ax^3 - \;\; x^4 \\
\hline
\qquad\qquad -2x^4 \\
\hline
\end{array}$$

EXAMPLES FOR PRACTICE.

1. Divide $a^2 + 4ax + 4x^2$ by $a + 2x$. Ans. $a + 2x$.
2. Divide $a^3 - 3a^2z + 3az^2 - z^3$ by $a - z$.
   Ans. $a^2 - 2az + z^2$.
3. Divide 1 by $1 + a$. Ans. $1 - a + a^2 - a^3 +$ &c.
4. Divide $12x^4 - 192$ by $3x - 6$.
   Ans. $4x^3 + 8x^2 + 16x + 32$.
5. Divide $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$ by $a^2 - 2ab + b^2$. Ans. $a^3 - 3a^2b + 3ab^2 - b^3$.
6. Divide $48z^3 - 96az^2 - 64a^2z + 150a^3$ by $2z - 3a$.
7. Divide $b^6 - 3b^4x^2 + 3b^2x^4 - x^6$ by $b^3 - 3b^2x + 3bx^2 - x^3$.
8. Divide $a^7 - x^7$ by $a - x$.
9. Divide $a^3 + 5a^2x + 5ax^2 + x^3$ by $a + x$.
10. Divide $a^4 + 4a^2b^2 - 32b^4$ by $a + 2b$.
11. Divide $24a^4 - b^4$ by $3a - 2b$.

ALGEBRAIC FRACTIONS.

Algebraic Fractions have the same names and rules of operation, as numeral fractions in common arithmetic; as [in] the following Rules and Cases.

CASE.

CASE I.

To Reduce a Mixed Quantity to an Improper Fraction.

MULTIPLY the integer by the denominator of the fraction, and to the product add the numerator, or connect it with its proper sign, + or −; then the denominator being set under this sum, will give the improper fraction required.

EXAMPLES.

1. Reduce $3\frac{4}{5}$, and $a-\frac{b}{x}$ to improper fractions.

First, $3\frac{4}{5}=\frac{3\times5+4}{5}=\frac{15+4}{5}=\frac{19}{5}$ the Answer.

And, $a-\frac{b}{x}=\frac{a\times x-b}{x}=\frac{ax-b}{x}$ the Answer.

2. Reduce $a+\frac{a^2}{b}$ and $a-\frac{z^2-a^2}{a}$ to improper fractions.

First, $a+\frac{a^2}{b}=\frac{a\times b+a^2}{b}=\frac{ab+a^2}{b}$ the Answer.

And, $a-\frac{z^2-a^2}{a}=\frac{a^2-z^2+a^2}{a}=\frac{2a^2-z^2}{a}$ the Answer.

3. Reduce $5\frac{3}{7}$ to an improper fraction. Ans. $\frac{38}{7}$.

4. Reduce $1-\frac{3a}{x}$ to an improper fraction. Ans. $\frac{x-3a}{x}$.

5. Reduce $2a-\frac{3ax+a^2}{4x}$ to an improper fraction.

6. Reduce $12+\frac{4x-18}{5x}$ to an improper fraction.

7. Reduce $x+\frac{1-3a-c}{c}$ to an improper fraction.

8. Reduce $4+2x-\frac{2x^3-3a}{5a}$ to an improper fraction.

CASE II.

To Reduce an Improper Fraction to a Whole or Mixed Quantity.

DIVIDE the numerator by the denominator, for the integral part; and set the remainder, if any, over the denominator, for the fractional part; the two joined together will be the mixed quantity required.

EXAMPLES.

1. To reduce $\frac{16}{3}$ and $\frac{ab+a^2}{b}$ to mixed quantities.

First, $\frac{16}{3} = 16 \div 3 = 5\frac{1}{3}$, the Answer required.

And, $\frac{ab+a^2}{b} = \overline{ab+a^2} \div b = a + \frac{a^2}{b}$. Answer.

2. To reduce $\frac{2ac-3a^2}{c}$ and $\frac{3ax+4x^2}{a+x}$ to mixed quantities.

First, $\frac{2ac-3a^2}{c} = \overline{2ac-3a^2} \div c = 2a - \frac{3a^2}{c}$. Answer.

And, $\frac{3ax+4x^2}{a+x} = \overline{3ax+4x^2} \div \overline{a+x} = 3x + \frac{x^2}{a+x}$. Ans.

3. Reduce $\frac{33}{5}$ and $\frac{2ax-3x^2}{a}$ to mixed quantities.

Ans. $6\frac{3}{5}$, and $2x - \frac{3x^2}{a}$.

4. Reduce $\frac{4a^2x}{2a}$ and $\frac{2a^2+2b^2}{a-b}$ to whole or mixed quantities.

5. Reduce $\frac{3x^2-3y^2}{x+y}$, and $\frac{2x^3-2y^3}{x-y}$ to whole or mixed quantities.

6. Reduce $\frac{10a^2-4a+6}{5a}$ to a mixed quantity.

7. Reduce $\frac{15a^3+5a^2}{3a^3+2a^2-2a-4}$ to a mixed quantity.

CASE III.

To Reduce Fractions to a Common Denominator.

MULTIPLY every numerator, separately, by all the denominators except its own, for the new numerators; and all the denominators together, for the common denominator.

When the denominators have a common divisor, it will be better, instead of multiplying by the whole denominators, to multiply only by those parts which arise from dividing by the common divisor. And observing also the several rules and directions as in Fractions in the Arithmetic.

EXAMPLES.

EXAMPLES.

1. Reduce $\frac{a}{x}$ and $\frac{b}{z}$ to a common denominator.

Here $\frac{a}{x}$ and $\frac{b}{z} = \frac{az}{xz}$ and $\frac{bx}{xz}$, by multiplying the terms of the first fraction by z, and the terms of the 2d by x.

2. Reduce $\frac{a}{x}$, $\frac{x}{b}$, and $\frac{b}{c}$ to a common denominator.

Here $\frac{a}{x}$, $\frac{x}{b}$, and $\frac{b}{c} = \frac{abc}{bcx}$, $\frac{cx^2}{bcx}$, and $\frac{b^2x}{bcx}$, by multiplying the terms of the 1st fraction by bc, of the 2d by cx, and of the 3d by bx.

3. Reduce $\frac{2a}{x}$ and $\frac{3b}{2c}$ to a common denominator.

Ans. $\frac{4ac}{2cx}$ and $\frac{3bx}{2cx}$.

4. Reduce $\frac{2a}{b}$ and $\frac{3a+2b}{2c}$ to a common denominator.

Ans. $\frac{4ac}{2bc}$, and $\frac{3ab+2b^2}{2bc}$.

5. Reduce $\frac{5a}{3x}$ and $\frac{3b}{2c}$, and $4d$, to a common denominator.

Ans. $\frac{10ac}{6cx}$ and $\frac{9bx}{6cx}$ and $\frac{24cdx}{6cx}$.

6. Reduce $\frac{5}{6}$ and $\frac{3a}{4}$ and $2b + \frac{3a}{b}$, to fractions having a common denominator. Ans. $\frac{20b}{24b}$ and $\frac{18ab}{24b}$, and $\frac{48b^2 + 72a}{24b}$.

7. Reduce $\frac{1}{3}$ and $\frac{2a^2}{4}$ and $\frac{2a^2+b^2}{a+b}$ to a common denominator.

8. Reduce $\frac{3b}{4a^2}$ and $\frac{2c}{3a}$ and $\frac{d}{2a}$ to a common denominator.

CASE

CASE IV.

To find the Greatest Common Measure of the Terms of a Fraction.

DIVIDE the greater term by the less, and the last divisor by the last remainder, and so on till nothing remains; then the divisor last used will be the common measure required; just the same as in common numbers.

But note, that it is proper to range the quantities according to the dimensions of some letters, as is shown in division. And note also, that all the letters or figures which are common to each term of the divisors, must be thrown out of them, or must divide them, before they are used in the operation:

EXAMPLES.

1. To find the greatest common measure of $\dfrac{ab+b^2}{ac^2+bc^2}$.

$$\begin{array}{rl}
ab+b^2\ ) & ac^2+bc^2 \\
\text{or } a+b\ ) & ac^2+bc^2\ (\ c^2 \\
 & \underline{ac^2+bc^2} \\
\end{array}$$

Therefore the greatest common measure is $a+b$.

2. To find the greatest common measure of $\dfrac{a^3-ab^2}{a^2+2ab+b^2}$.

$$\begin{array}{l}
a^2+2ab+b^2\ )\ a^3-ab^2\ (\ a \\
\qquad\qquad\qquad \underline{a^3+2a^2b+ab^2} \\
\qquad\qquad\qquad -2a^2b-2ab^2\ )\ a^2+2ab+b^2 \\
\qquad\qquad\qquad \text{or}\quad a+b\ )\ a^2+2ab+b^2\ (\ a+b. \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{a^2+ab} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad ab+b^2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{ab+b^2}
\end{array}$$

Therefore $a+b$ is the greatest common divisor.

3. To find the greatest common divisor of $\dfrac{a^2-4}{ab+2b}$.

Ans. $a-2$.

4. To

4. To find the greatest common divisor of $\frac{a^5 - a^3b^2}{a^4 - b^4}$.

Ans. $a^2 - b^2$.

5. Find the greatest com. measure of $\frac{a^3x + 2a^2x^2 + 2ax^3 + x^4}{5a^5 + 10a^4x + 5a^3x^2}$.

CASE V.

To Reduce a Fraction to its Lowest Terms.

FIND the greatest common measure, as in the last problem. Then divide both the terms of the fraction by the common measure thus found, and it will reduce it to its lowest terms at once, as was required. Or divide the terms by any quantity which it may appear will divide them both, as in arithmetical fractions.

EXAMPLES.

1. Reduce $\frac{ab + b^2}{ac^2 + bc^2}$ to its lowest terms.

$$\begin{array}{rl} ab + b^2\ ) & ac^2 + bc^2 \\ \text{or } a + b\ ) & ac^2 + bc^2\ (c^2 \\ & \underline{ac^2 + b^2c^2} \end{array}$$

Here $ab + b^2$ is divided by the common factor b.

Therefore $a + b$ is the greatest common measure, and hence $a + b\ )\ \frac{ab + b^2}{ac^2 + bc^2} = \frac{b}{c^2}$, is the fraction required.

2. To reduce $\frac{c^3 - b^2c}{c^2 + 2bc + b^2}$ to its least terms.

$$\begin{array}{rl} c^2 + 2bc + b^2\ ) & c^3 - b^2c\ (c \\ & \underline{c^3 + 2bc^2 + b^2c} \\ & -2bc^2 - 2b^2c\ )\ c^2 + 2bc + b^2 \\ & \text{or } c + b\ )\ c^2 + 2bc + b^2\ (c + b \\ & \qquad\qquad \underline{c^2 + bc} \\ & \qquad\qquad\quad bc + b^2 \\ & \qquad\qquad\quad \underline{bc + b^2} \end{array}$$

Therefore

Therefore $c+b$ is the greatest common measure, and hence $c+b)\dfrac{c^3-b^2c}{c^2+2bc+b^2}=\dfrac{c^2-bc}{c+b}$ is the fraction required.

3. Reduce $\dfrac{c^3-b^3}{c^4-b^2c^2}$ to its lowest terms. Ans. $\dfrac{c^2+bc+b^2}{c^3+bc^2}$.

4. Reduce $\dfrac{a^2-b^2}{a^4-b^4}$ to its lowest terms. Ans. $\dfrac{1}{a^2+b^2}$.

5. Reduce $\dfrac{a^4-b^4}{a^3-3a^2b+3ab^2-b^3}$ to its lowest terms.

6. Reduce $\dfrac{3a^5+6a^4c+3a^3c^2}{a^3c+3a^2c^2+3ac^3+c^4}$ to its lowest terms.

7. Reduce $\dfrac{a^3-ab^2}{a^2+2ab+b^2}$ to its lowest terms.

CASE VI.

To add Fractional Quantities together.

If the fractions have a common denominator, add all the numerators together; then under their sum set the common denominator, and it is done.

If they have not a common denominator, reduce them to one, and then add them as before.

EXAMPLES.

1. Let $\dfrac{a}{3}$ and $\dfrac{a}{4}$ be given, to find their sum.

Here $\dfrac{a}{3}+\dfrac{a}{4}=\dfrac{4a}{12}+\dfrac{3a}{12}=\dfrac{7a}{12}$ is the sum required.

2. Given $\dfrac{a}{b}$, $\dfrac{b}{c}$, and $\dfrac{c}{d}$, to find their sum.

Here $\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{d}=\dfrac{acd}{bcd}+\dfrac{bbd}{bcd}+\dfrac{bcc}{bcd}=\dfrac{acd+bbd+bcc}{bcd}$ the sum required.

3. Let

* 3. Let $a - \frac{3x^2}{b}$ and $b + \frac{2ax}{c}$ be added together.

Here $a - \frac{3x^2}{b} + b + \frac{2ax}{c} = a - \frac{3cx^2}{bc} + b + \frac{2abx}{bc}$

$= a + b + \frac{2abx - 3cx^2}{bc}$, the sum required.

4. Add $\frac{4x}{3a}$ and $\frac{2x}{5b}$ together. Ans. $\frac{20bx + 6ax}{15ab}$.

5. Add $\frac{a}{3}$, $\frac{a}{4}$ and $\frac{a}{5}$ together. Ans. $\frac{47}{60}a$.

6. Add $\frac{2a-3}{4}$ and $\frac{5a}{8}$ together. Ans. $\frac{9a-6}{8}$.

7. Add $2a + \frac{a+3}{5}$ to $4a + \frac{2a-5}{4}$. Ans. $6a + \frac{14a-13}{20}$.

8. Add $6a$, and $\frac{3a^2}{4b}$ and $\frac{a+b}{3b}$ together.

9. Add $\frac{5a}{4}$, and $\frac{6a}{5}$ and $\frac{3a+2}{7}$ together.

10. Add $2a$, and $\frac{3a}{8}$ and $3 + \frac{a}{6}$ together.

11. Add $8a + \frac{3a}{4}$ and $2a - \frac{5a}{8}$ together.

CASE VII.

* *To Subtract one Fractional Quantity from another.*

Reduce the fractions to a common denominator, as in addition, if they have not a common denominator.

Subtract the numerators from each other, and under their difference set the common denominator, and the work is done.

---

* In the addition of mixed quantities, it is best to bring the fractional parts only to a common denominator, and to annex their sum to the sum of the integers, with the proper sign. And the same rule may be observed for mixed quantities in subtraction also.

EXAMPLES.

EXAMPLES.

1. To find the difference of $\frac{3a}{4}$ and $\frac{4a}{7}$.

Here $\frac{3a}{4}-\frac{4a}{7}=\frac{21a}{28}-\frac{16a}{28}=\frac{5a}{28}$ is the difference required.

2. To find the difference of $\frac{2a-b}{4c}$ and $\frac{3a-4b}{3b}$.

Here $\frac{2a-b}{4c}-\frac{3a-4b}{3b}=\frac{6ab-3bb}{12bc}-\frac{12ac-16bc}{12bc}=$

$\frac{6ab-3bb-12ac+16bc}{12bc}$ is the difference required.

3. Required the difference of $\frac{10a}{9}$ and $\frac{4a}{7}$.

4. Required the difference of $6a$ and $\frac{3a}{4}$.

5. Required the difference of $\frac{5a}{4}$ and $\frac{2a}{3}$.

6. Subtract $\frac{2b}{c}$ from $\frac{3a+c}{b}$.

7. Take $\frac{2a+6}{9}$ from $\frac{4a+8}{5}$.

8. Take $2a-\frac{a-3b}{c}$ from $4a+\frac{2a}{c}$.

CASE VIII.

To Multiply Fractional Quantities together.

MULTIPLY the numerators together for a new numerator, and the denominators for a new denominator *.

---

* 1. When the numerator of one fraction, and the denominator of the other, can be divided by some quantity, which is common to both, the quotients may be used instead of them.

2. When a fraction is to be multiplied by an integer, the product is found either by multiplying the numerator, or dividing the denominator by it; and if the integer be the same with the denominator, the numerator may be taken for the product.

EXAMPLES.

EXAMPLES.

1. Required to find the product of $\frac{a}{8}$ and $\frac{2a}{5}$.

Here $\frac{a \times 2a}{8 \times 5} = \frac{2a^2}{40} = \frac{a^2}{20}$ the product required.

2. Required the product of $\frac{a}{3}$, $\frac{3a}{4}$, and $\frac{6a}{7}$.

$\frac{a \times 3a \times 6a}{3 \times 4 \times 7} = \frac{18a^3}{84} = \frac{3a^3}{14}$ the product required.

3. Required the product of $\frac{2a}{b}$ and $\frac{a+b}{2a+c}$.

Here $\frac{2a \times (a+b)}{b \times (2a+c)} = \frac{2aa+2ab}{2ab+bc}$ the product required.

4. Required the product of $\frac{4a}{3}$ and $\frac{6a}{5c}$.

5. Required the product of $\frac{3a}{4}$ and $\frac{4b^2}{3a}$.

6. To multiply $\frac{3a}{b}$, and $\frac{8ac}{b}$, and $\frac{4ab}{3c}$ together.

7. Required the product of $2a+\frac{ab}{2c}$ and $\frac{3a^2}{b}$.

8. Required the product of $\frac{2a^2-2b^2}{3bc}$ and $\frac{4a^2+2b^2}{a+b}$.

9. Required the product of $3a$, and $\frac{2a+1}{a}$, and $\frac{2a-1}{2a+b}$.

10. Multiply $a+\frac{x}{2a}-\frac{x^2}{4a^2}$ by $x-\frac{a}{2x}+\frac{a^2}{4x^2}$.

CASE IX.

To Divide one Fractional Quantity by another.

DIVIDE the numerators by each other, and the denominators by each other, if they will exactly divide. But, if not, then invert the terms of the divisor, and multiply by it exactly as in multiplication *.

EXAMPLES.

* 1. If the fractions to be divided have a common denominator, take the numerator of the dividend for a new numerator, and the numerator of the divisor for the new denominator.

2. When

EXAMPLES.

1. Required to divide $\frac{a}{4}$ by $\frac{3a}{8}$.

Here $\frac{a}{4} \div \frac{3a}{8} = \frac{a}{4} \times \frac{8}{3a} = \frac{8a}{12a} = \frac{2}{3}$ the quotient.

2. Required to dividé $\frac{3a}{2b}$ by $\frac{5c}{4d}$.

Here $\frac{3a}{2b} \div \frac{5c}{4d} = \frac{3a}{2b} \times \frac{4d}{5c} = \frac{12ad}{10bc} = \frac{6ad}{5bc}$ the quotient.

3. To divide $\frac{2a+b}{3a-2b}$ by $\frac{3a+2b}{4a+2b}$. Here,

$\frac{2a+b}{3a-2b} \times \frac{4a+b}{3a+2b} = \frac{8a^2+6ab+b^2}{9a^2-4b^2}$ the quotient required.

4. To divide $\frac{3a^2}{a^2+b^2}$ by $\frac{2a}{2a+2b}$.

Here, $\frac{3a^2}{a^2+b^3} \times \frac{a+b}{a} = \frac{3a^2 \times (a+b)}{(a^3+b^3) \times a} = \frac{3a}{a^2-ab+b^2}$

is the quotient required.

5. To divide $\frac{3x}{4}$ by $\frac{11}{12}$.

6. To divide $\frac{6x^2}{5}$ by $3x$.

7. To divide $\frac{3x+1}{9}$ by $\frac{4x}{3}$.

8. To divide $\frac{4x}{2x-1}$ by $\frac{x}{3}$.

9. To divide $\frac{4x}{5}$ by $\frac{3a}{5b}$.

10. To divide $\frac{2a-b}{4cd}$ by $\frac{5ac}{6d}$.

11. Divide $\frac{5a^4-5b^4}{2a^2-4ab+2b^2}$ by $\frac{6a^2+5ab}{4a-4b}$.

---

2. When a fraction is to be divided by any quantity, it is the same thing whether the numerator be divided by it, or the denominator multiplied by it.

3. When the two numerators, or the two denominators, can be divided by some common quantity, let that be done, and the quotients used instead of the fractions first proposed.

INVOLUTION.

INVOLUTION.

INVOLUTION is the raising of powers from any proposed root; such as finding the square, cube, biquadrate, &c, of any given quantity. The method is as follows:

* MULTIPLY the root or given quantity by itself, as many times as there are units in the index less one, and the last product will be the power required.—Or, in literals, multiply the index of the root by the index of the power, and the result will be the power, the same as before.

Note. When the sign of the root is $+$, all the powers of it will be $+$; but when the sign is $-$, all the even powers will be $+$, and all the odd powers $-$; as is evident from multiplication.

EXAMPLES.

a, the root
$a^2 =$ square
$a^3 =$ cube
$a^4 =$ 4th power
$a^5 =$ 5th power
&c.

$-\ 2a$, the root
$+\ 4a^2 =$ square
$-\ 8a^3 =$ cube
$+\ 16a^4 =$ 4th power
$-\ 32a^5 =$ 5th power

$-\ \frac{2ax^2}{3b}$, the root
$+\ \frac{4a^2x^4}{9b^2} =$ square
$-\ \frac{8a^3x^6}{27b^3} =$ cube
$+\ \frac{16a^4x^8}{81b^4} =$ 4th power.

a^2, the root
$a^4 =$ square
$a^6 =$ cube
$a^8 =$ 4th power
$a^{10} =$ 5th power
&c.

$-\ 3ab^2$, the root
$+\ 9a^2b^4 =$ square
$-\ 27a^3b^6 =$ cube
$+\ 81a^4b^8 =$ 4th power.
$-\ 243a^5b^{10} =$ 5th power.

$\frac{a}{2b}$, the root
$\frac{a^2}{4b^2} =$ square
$\frac{a^3}{8b^3} =$ cube
$\frac{a^4}{16b^4} =$ biquadrate

* Any power of the product of two or more quantities, is equal to the same power of each of the factors, multiplied together.

And any power of a fraction, is equal to the same power of the numerator, divided by the like power of the denominator.

Also, powers or roots of the same quantity, are multiplied by one another, by adding their exponents; or divided, by subtracting their exponents.

Thus, $a^3 \times a^2 = a^{3+2} = a^5$. And $a^3 \div a^2$ or $\frac{a^3}{a^2} = a^{3-2} = a$.

x — a.

$$\begin{array}{ll}
x-a = \text{root} & x+a = \text{root} \\
x-a & x+a \\
\hline
x^2-ax & x^2+ax \\
\quad -ax+a^2 & \quad +ax+a^2 \\
\hline
x^2-2ax+a^2 \text{ square} & x^2+2ax+a^2 \\
x-a & x+a \\
\hline
x^3-2ax^2+a^2x & x^3+2ax^2+a^2x \\
\quad -ax^2+2a^2x-a^3 & \quad +ax^2+2a^2x+a^3 \\
\hline
x^3-3ax^2+3a^2x-a^3 & x^3+3ax^2+3a^2x+a^3 \\
\hline
\end{array}$$

the cubes, or third powers, of $x-a$ and $x+a$.

EXAMPLES FOR PRACTICE.

1. Required the cube or 3d power of $3a^2$.
2. Required the 4th power of $2a^2b$.
3. Required the 3d power of $-4a^2b^3$.
4. To find the biquadrate of $-\dfrac{a^2x}{2b^2}$.
5. Required the 5th power of $a-2x$.
6. To find the 6th power of $2a^{\frac{1}{2}}$.

SIR ISAAC NEWTON'S RULE *for raising a Binomial to any Power whatever* *.

1. *To find the Terms without the Co-efficients.* The index of the first, or leading quantity, begins with the index of the given power, and in the succeeding terms decreases continually by 1, in every term to the last; and in the 2d or following quantity, the indices of the terms are 0, 1, 2, 3, 4, &c, increasing always by 1. That is, the first term will contain only the 1st part of the root with the same index, or of

* This rule, expressed in general terms, is as follows:

$$(a+x)^n = a^n + n.a^{n-1}x + n.\frac{n-1}{2}a^{n-2}x^2 + n.\frac{n-1}{2}.\frac{n-2}{3}a^{n-3}x^3\ \&c.$$

$$(a-x)^n = a^n - n.a^{n-1}x + n.\frac{n-1}{2}a^{n-2}x^2 - n.\frac{n-1}{2}.\frac{n-2}{3}a^{n-3}x^3\ \&c.$$

Note. The sum of the co-efficients, in every power, is equal to the number 2, when raised to that power. Thus $1+1=2$ in the first power; $1+2+1=4=2^2$ in the square; $1+3+3+1=8=2^3$ in the cube, or third power; and so on.

the

the same height as the intended power: and the last term of the series will contain only the 2d part of the given root, when raised also to the same height of the intended power: but all the other or intermediate terms will contain the products of some powers of both the members of the root, in such sort, that the powers or indices of the 1st or leading member will always decrease by 1, while those of the 2d member always increase by 1.

2. *To find the Co-efficients.* The first co-efficient is always 1, and the second is the same as the index of the intended power; to find the 3d co-efficient, multiply that of the 2d term by the index of the leading letter in the same term, and divide the product by 2; and so on, that is, multiply the co-efficient of the term last found by the index of the leading quantity in that term, and divide the product by the number of terms to that place, and it will give the co-efficient of the term next following; which rule will find all the co-efficients, one after another.

Note. The whole number of terms will be 1 more than the index of the given power: and when both terms of the root are $+$, all the terms of the power will be $+$; but if the second term be $-$, all the odd terms will be $+$, and all the even terms $-$, which causes the terms to be $+$ and $-$ alternately. Also the sum of the two indices, in each term, is always the same number, viz. the index of the required power: and, counting from the middle of the series, both ways, or towards the right and left, the indices of the two terms are the same figures at equal distances, but mutually changed places. Moreover, the co-efficients are the same numbers at equal distances from the middle of the series, towards the right and left; so by whatever numbers they increase to the middle, by the same in the reverse order they decrease to the end.

EXAMPLES.

1. Let $a + x$ be involved to the 5th power.

The terms without the co-efficients, by the 1st rule, will be

$$a^5,\ a^4x,\ a^3x^2,\ a^2x^3,\ ax^4,\ x^5,$$

and the co-efficients, by the 2d rule, will be

$$1,\ 5,\ \frac{5\times4}{2},\ \frac{10\times3}{3},\ \frac{10\times2}{4},\ \frac{5\times1}{5};$$

$$\text{or, } 1,\ 5,\ 10,\ \ 10,\ \ 5,\ \ 1;$$

Therefore the 5th power altogether is

$$a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5.$$

But

But it is best to set down both the co-efficients and the powers of the letters at once, in one line, without the intermediate lines in the above example, as in the example here below.

2. Let $a-x$ be involved to the 6th power.

The terms with the co-efficients will be

$a^6-6a^5x+15a^4x^2-20a^3x^3+15a^2x^4-6ax^5+x^6$.

3. Required the 4th power of $a-x$.

Ans. $a^4-4a^3x+6a^2x^2-4ax^3+x^4$.

And thus any other powers may be set down at once, in the same manner; which is the best way.

EVOLUTION.

EVOLUTION is the reverse of Involution, being the method of finding the square root, cube root, &c, of any given quantity, whether simple or compound.

CASE I. *To find the Roots of Simple Quantities.*

EXTRACT the root of the co-efficient, for the numeral part; and divide the index of the letter or letters, by the index of the power, and it will give the root of the literal part; then annex this to the former, for the whole root sought*.

---

* Any even root of an affirmative quantity, may be either $+$ or $-$: thus the square root of $+a^2$ is either $+a$, or $-a$; because $+a\times+a=+a^2$, and $-a\times-a=+a^2$ also.

But an odd root of any quantity will have the same sign as the quantity itself: thus the cube root of $+a^3$ is $+a$, and the cube root of $-a^3$ is $-a$; for $+a\times+a\times+a=+a^3$, and $-a\times-a\times-a=-a^3$.

Any even root of a negative quantity is impossible; for neither $+a\times+a$, nor $-a\times-a$ can produce $-a^2$.

Any root of a product, is equal to the like root of each of the factors multiplied together. And for the root of a fraction, take the root of the numerator, and the root of the denominator.

EXAMPLES.

EXAMPLES.

1. The square root of $4a^2$, is $2a$.

2. The cube root of $8a^3$, is $2a^{\frac{3}{3}}$ or $2a$.

3. The square root of $\frac{5a^2b^2}{9c^2}$, or $\sqrt{\frac{5a^2b^2}{9c^2}}$, is $\frac{ab}{3c}\sqrt{5}$.

4. The cube root of $-\frac{16a^4b^6}{27c^3}$, is $-\frac{2ab^2}{3c}\sqrt[3]{2a}$.

5. To find the square root of $2a^2b^4$. Ans. $ab^2\sqrt{2}$.

6. To find the cube root of $-64a^3b^6$. Ans. $-4ab^2$.

7. To find the square root of $\frac{8a^2b^2}{3c^3}$. Ans. $2ab\sqrt{\frac{2}{3c}}$.

8. To find the 4th root of $81a^4b^6$. Ans. $3ab\sqrt{b}$.

9. To find the 5th root of $-32a^5b^6$. Ans. $-2ab\sqrt[5]{b}$.

CASE II.

To find the Square Root of a Compound Quantity.

This is performed like as in numbers, thus:

1. Range the quantities according to the dimensions of one of the letters, and set the root of the first term in the quotient.

2. Subtract the square of the root, thus found, from the first term, and bring down the next two terms to the remainder for a dividend; and take double the root for a divisor.

3. Divide the dividend by the divisor, and annex the result both to the quotient and to the divisor.

4. Multiply the divisor, thus increased, by the term last set in the quotient, and subtract the product from the dividend.

And so on, always the same, as in common arithmetic.

EXAMPLES.

1. Extract the square root of $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$.

$$a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 \;(\, a^2 - 2ab + b^2 \text{ the root.}$$
$$a^4$$
$$2a^2 - 2ab)\; -4a^3b + 6a^2b^2$$
$$-4a^3b + 4a^2b^2$$
$$2a^2 - 4ab + b^2)\; 2a^2b^2 - 4ab^3 + b^4$$
$$2a^2b^2 - 4ab^3 + b^4$$

2. Find

2. Find the root of $a^4 + 4a^3b + 10a^2b^2 + 12ab^3 + b^4$.

$$\begin{array}{l} a^4 + 4a^3b + 10a^2b^2 + 12ab^3 + b^4 \,(\, a^2 + 2ab + 3b^2. \\ a^4 \\ \hline 2a^2 + 2ab \,)\; 4a^3b + 10a^2b^2 \\ \qquad\qquad\quad 4a^3b + \;\; 4a^2b^2 \\ \hline 2a^2 + 4ab + 3b^2 \,)\; 6a^2b^2 + 12ab^3 + b^4 \\ \qquad\qquad\qquad\qquad 6a^2b^2 + 12ab^3 + b^4 \\ \hline \end{array}$$

3. To find the square root of $a^4 + 4a^3 + 6a^2 + 4a + 1$.
Ans. $a^2 + 2a + 1$.

4. Extract the square root of $a^4 - 2a^3 + 2a^2 - a + \frac{1}{4}$.
Ans. $x^2 - x + \frac{1}{2}$.

5. It is required to find the square root of $a^2 - ab$.
Ans. $a - \frac{b}{2} - \frac{b^2}{8a} - \frac{b^3}{16a^2} -$ &c.

CASE III.

To find the Roots of any Powers in General.

THIS is also done like the same roots in numbers, thus:

Find the root of the first term, and set it in the quotient. —Subtract its power from that term, and bring down the second term for a dividend.—Involve the root, last found, to the next lower power, and multiply it by the index of the given power, for a divisor.—Divide the dividend by the divisor, and set the quotient as the next term of the root.—Involve now the whole root to the power to be extracted; then subtract the power thus arising from the given power, and divide the first term of the remainder by the divisor first found; and so on till the whole is finished *.

EXAMPLES.

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* As this method, in high powers, may be thought too laborious, it will not be improper to observe, that the roots of compound quantities may sometimes be easily discovered, thus:

Extract the roots of some of the most simple terms, and connect them together by the sign + or −, as may be judged most suitable for the purpose.—Involve the compound root, thus found, to the proper power; then, if this be the same with the given quantity, it is the root required.—But if it be found to differ only in some of the signs, change them from + to −, or from − to +, till its power agrees with the given one throughout.

Thus,

EXAMPLES.

1. To find the square root of $a^4-2a^3b+3a^2b^2-2ab^3+b^4$.

$$a^4-2a^3b+3a^2b^2-2ab^3+b^4\ (\,a^2-ab+b^2$$
$$a^4$$
$$2a^2\,)\ -2a^3b$$
$$a^4-2a^3b+a^2b^2=(a^2-ab)^2$$
$$2a^2\,)\ 2a^2b^2$$
$$a^4-2a^3b+3a^2b^2-2ab^3+b^4=(a^2-ab+b^2)^2.$$

2. Find the cube root of $a^6-6a^5+21a^4-44a^3+63a^2-54a+27$.

$$a^6-6a^5+21a^4-44a^3+63a^2-54a+27\ (\,a^2-2a+3.$$
$$a^6$$
$$3a^4\,)-6a^5$$
$$a^6-6a^5+12a^4-8a^3=(a^2-2a)^3$$
$$3a^4\,)+12a^4$$
$$a^6-6x^5+21a^4-44a^3+63a^2-54a+27=(a^2-2a-3)^3.$$

3. To find the square root of $a^2-2ab+2ax+b^2-2bx+x^2$. Ans. $a-b+x$.

4. Find the cube root of $a^6-3a^5+9a^4-13a^3+18a^2-12a+8$. Ans. a^2-a+2.

5. Find the 4th root of $81a^4-216a^3b+216a^2b^2-96ab^3+16b^4$. Ans. $3a-2b$.

6. Find the 5th root of $a^5-10a^4+40a^3-80a^2+80a-32$. Ans. $a-2$.

7. Required the square root of $1-x^2$.

8. Required the cube root of $1-x^3$.

Thus, in the 5th example, the root $3a-2b$, is the difference of the roots of the first and last terms; and in the 3d example, the root $a-b+x$, is the sum of the roots of the 1st, 4th, and 6th terms. The same may also be observed of the 6th example, where the root is found from the first and last terms.

SURDS.

SURDS are such quantities as have no exact root; and are usually expressed by fractional indices, or by means of the radical sign $\sqrt{}$. Thus, $3^{\frac{1}{2}}$, or $\sqrt{3}$, denotes the square root of 3; and $2^{\frac{2}{3}}$ or $\sqrt[3]{2^2}$, or $\sqrt[3]{4}$, the cube root of the square of 2; where the numerator shows the power to which the quantity is to be raised, and the denominator its root.

PROBLEM I.

To Reduce a Rational Quantity to the Form of a Surd.

RAISE the given quantity to the power denoted by the index of the surd; then over or before this new quantity set the radical sign, and it will be of the form required.

EXAMPLES.

1. To reduce 4 to the form of the square root.
First, $4^2 = 4 \times 4 = 16$; then $\sqrt{16}$ is the answer.

2. To reduce $3a^2$ to the form of the cube root.
First, $3a^2 \times 3a^2 \times 3a^2 = (3a^2)^3 = 27a^6$;
then $\sqrt[3]{27a^6}$ or $(27a^6)^{\frac{1}{3}}$ is the answer.

3. Reduce 6 to the form of the cube root.
Ans. $(216)^{\frac{1}{3}}$ or $\sqrt[3]{216}$.

4. Reduce $\frac{1}{3}ab$ to the form of the square root.
Ans. $\sqrt{\frac{1}{9}a^2b^2}$.

5. Reduce 2 to the form of the 4th root. Ans. $(16)^{\frac{1}{4}}$.

6. Reduce $a^{\frac{1}{3}}$ to the form of the 5th root.

7. Reduce $a+x$ to the form of the square root.

8. Reduce $a-x$ to the form of the cube root.

PROBLEM II.

To Reduce Quantities to a Common Index.

1. REDUCE the indices of the given quantities to a common denominator, and involve each of them to the power denoted by its numerator; then 1 set over the common denominator will form the common index. Or,

2. If

2. If the common index be given, divide the indices of the quantities by the given index, and the quotients will be the new indices for those quantities. Then over the said quantities, with their new indices, set the given index, and they will make the equivalent quantities sought.

EXAMPLES.

1. Reduce $3^{\frac{1}{2}}$ and $5^{\frac{1}{5}}$ to a common index.
Here $\frac{1}{2}$ and $\frac{1}{5} = \frac{5}{10}$ and $\frac{2}{10}$.
Therefore $3^{\frac{5}{10}}$ and $5^{\frac{2}{10}} = (3^5)^{\frac{1}{10}}$ and $(5^2)^{\frac{1}{10}} = \sqrt[10]{3^5}$ and $\sqrt[10]{5^2}$ $= \sqrt[10]{243}$ and $\sqrt[10]{25}$.

2. Reduce a^3 and $b^{\frac{1}{3}}$ to the same common index $\frac{1}{2}$.
Here, $\frac{3}{1} \div \frac{1}{2} = \frac{3}{1} \times \frac{2}{1} = \frac{6}{1}$ the 1st index,
and $\frac{1}{3} \div \frac{1}{2} = \frac{1}{3} \times \frac{2}{1} = \frac{2}{3}$ the 2d index.
Therefore $(a^6)^{\frac{1}{2}}$ and $(b^{\frac{2}{3}})^{\frac{1}{2}}$, or $\sqrt{a^6}$ and $\sqrt{b^{\frac{2}{3}}}$ are the quantities.

3. Reduce $4^{\frac{1}{3}}$ and $5^{\frac{1}{2}}$ to the common index $\frac{1}{4}$.

Ans. $256^{\frac{1}{3}})^{\frac{1}{4}}$ and $25^{\frac{1}{4}}$.

4. Reduce $a^{\frac{1}{3}}$ and $x^{\frac{1}{4}}$ to the common index $\frac{1}{6}$.

Ans. $(a^2)^{\frac{1}{6}}$ and $(x^{\frac{3}{2}})^{\frac{1}{6}}$.

5. Reduce a^2 and x^3 to the same radical sign.

Ans. $\sqrt{a^4}$ and $\sqrt{x^6}$.

6. Reduce $(a + x)^{\frac{1}{3}}$ and $(a - x)^{\frac{1}{2}}$ to a common index.

7. Reduce $(a + b)^{\frac{1}{2}}$ and $(a - b)^{\frac{1}{4}}$ to a common index.

PROBLEM III.

To Reduce Surds to more Simple Terms.

Find out the greatest power contained in, or to divide the given surd; take its root, and set it before the quotient or the remaining quantities, with the proper radical sign between them.

EXAMPLES.

1. To reduce $\sqrt{32}$ to simpler terms.
Here $\sqrt{32} = \sqrt{16 \times 2} = \sqrt{16} \times \sqrt{2} = 4 \times \sqrt{2} = 4\sqrt{2}$.

2. To reduce $\sqrt[3]{320}$ to simpler terms.
$\sqrt[3]{320} = \sqrt[3]{64 \times 5} = \sqrt[3]{64} \times \sqrt[3]{5} = 4 \times \sqrt[3]{5} = 4\sqrt[3]{5}$.

3. Reduc

3. Reduce $\sqrt{75}$ to its simplest terms. Ans. $5\sqrt{3}$.

4. Reduce $\sqrt{\frac{44}{75}}$ to simpler terms. Ans. $\frac{2}{15}\sqrt{33}$.

5. Reduce $\sqrt[3]{189}$ to its simplest terms. Ans. $3\sqrt[3]{7}$.

6. Reduce $\sqrt[3]{\frac{135}{32}}$ to its simplest terms. Ans. $\frac{3}{4}\sqrt[3]{10}$.

7. Reduce $\sqrt{75a^2b}$ to its simplest terms. Ans. $5a\sqrt{3b}$.

Note. There are other cases of reducing algebraic surds to simpler forms, that are practised on several occasions; one instance of which, on account of its simplicity and usefulness, may be here noticed, viz. in fractional forms having compound surds in the denominator, multiply both numerator and denominator by the same terms of the denominator, but having one sign changed, from + to − or from − to +, which will reduce the fraction to a rational denominator.

Ex. To reduce $\frac{\sqrt{20}+\sqrt{12}}{\sqrt{5}-\sqrt{3}}$, multiply it by $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$, and it becomes $\frac{16+2\sqrt{15}}{2}=8+\sqrt{15}$. Also, if $\frac{3\sqrt{15}-4\sqrt{5}}{\sqrt{15}+\sqrt{5}}$; multiply it by $\frac{\sqrt{15}-\sqrt{5}}{\sqrt{15}-\sqrt{5}}$, and it becomes $\frac{65-7\sqrt{75}}{15-5}=\frac{65-35\sqrt{3}}{10}=\frac{13-7\sqrt{3}}{2}$.

PROBLEM IV.

To add Surd Quantities together.

1. BRING all fractions to a common denominator, and reduce the quantities to their simplest terms, as in the last problem.—2. Reduce also such quantities as have unlike indices to other equivalent ones, having a common index.—3. Then, if the surd part be the same in them all, annex it to the sum of the rational parts, with the sign of multiplication, and it will give the total sum required.

But if the surd part be not the same in all the quantities, they can only be added by the signs + and −.

EXAMPLES.

1. Required to add $\sqrt{18}$ and $\sqrt{32}$ together.

First, $\sqrt{18}=\sqrt{9\times 2}=3\sqrt{2}$; and $\sqrt{32}=\sqrt{16\times 2}=4\sqrt{2}$: Then, $3\sqrt{2}+4\sqrt{2}=(3+4)\sqrt{2}=7\sqrt{2}$ = sum required.

2. It is required to add $\sqrt[3]{375}$, and $\sqrt[3]{192}$ together.

First, $\sqrt[3]{375}=\sqrt[3]{125\times 3}=5\sqrt[3]{3}$; and $\sqrt[3]{192}=\sqrt[3]{64\times 3}=4$
en, $5\sqrt[3]{3}+4\sqrt[3]{3}=(5+4)\sqrt[3]{3}=9\sqrt[3]{3}$ = sum req

3.

3. Required the sum of $\sqrt{27}$ and $\sqrt{48}$. Ans. $7\sqrt{3}$.

4. Required the sum of $\sqrt{50}$ and $\sqrt{72}$. Ans. $11\sqrt{2}$.

5. Required the sum of $\sqrt{\frac{3}{5}}$ and $\sqrt{\frac{1}{15}}$.
Ans. $4\sqrt{\frac{1}{15}}$ or $\frac{4}{15}\sqrt{15}$.

6. Required the sum of $\sqrt[3]{56}$ and $\sqrt[3]{189}$. Ans. $5\sqrt[3]{7}$.

7. Required the sum of $\sqrt[3]{\frac{1}{4}}$ and $\sqrt[3]{\frac{1}{32}}$. Ans. $\frac{3}{4}\sqrt[3]{2}$.

8. Required the sum of $3\sqrt{a^2b}$ and $5\sqrt{16a^4b}$.

PROBLEM V.

To find the Difference of Surd Quantities.

PREPARE the quantities the same way as in the last rule; then subtract the rational parts, and to the remainder annex the common surd, for the difference of the surds required.

But if the quantities have no common surd, they can only be subtracted by means of the sign $-$.

EXAMPLES.

1. To find the difference between $\sqrt{320}$ and $\sqrt{80}$.

First, $\sqrt{320}=\sqrt{64\times 5}=8\sqrt{5}$; and $\sqrt{80}=\sqrt{16\times 5}=4\sqrt{5}$.
Then $8\sqrt{5}-4\sqrt{5}=4\sqrt{5}$ the difference sought.

2. To find the difference between $\sqrt[3]{128}$ and $\sqrt[3]{54}$.

First, $\sqrt[3]{128}=\sqrt[3]{64\times 2}=4\sqrt[3]{2}$; and $\sqrt[3]{54}=\sqrt[3]{27\times 2}=3\sqrt[3]{2}$.
Then $4\sqrt[3]{2}-3\sqrt[3]{2}=\sqrt[3]{2}$, the difference required.

3. Required the difference of $\sqrt{75}$ and $\sqrt{48}$. Ans. $\sqrt{3}$.

4. Required the difference of $\sqrt[3]{256}$ and $\sqrt[3]{32}$. Ans. $2\sqrt[3]{4}$.

5. Required the difference of $\sqrt{\frac{3}{4}}$ and $\sqrt{\frac{2}{9}}$. Ans. $\frac{1}{6}\sqrt{6}$.

6. Required the difference of $\sqrt[3]{\frac{3}{5}}$ and $\sqrt[3]{\frac{2}{9}}$. Ans. $\frac{4}{15}\sqrt[3]{75}$.

7. Find the difference of $\sqrt{24a^2b^2}$ and $\sqrt{54ab^4}$.
Ans. $(a-2b)\surd(3b^2-2ab)\sqrt{6a}$.

PROBLEM VI.

To Multiply Surd Quantities together.

REDUCE the surds to the same index, if necessary; next mu[illegible] ...ntities together, and the surds toge- ...roduct to the other for the whole ...may be reduced to more simple

EXAMPLES.

EXAMPLES.

1. Required to find the product of $4\sqrt{12}$ and $3\sqrt{2}$.

Here, $4 \times 3 \times \sqrt{12} \times \sqrt{2} = 12\sqrt{12 \times 2} = 12\sqrt{24} = 12\sqrt{4 \times 6}$ $= 12 \times 2 \times \sqrt{6} = 24\sqrt{6}$, the product required.

2. Required to multiply $\frac{1}{4}\sqrt[3]{\frac{3}{4}}$ by $\frac{1}{3}\sqrt[3]{\frac{3}{8}}$.

Here $\frac{1}{4} \times \frac{1}{3}\sqrt[3]{\frac{3}{4}} \times \sqrt[3]{\frac{3}{8}} = \frac{1}{12} \times \sqrt[3]{\frac{9}{32}} = \frac{1}{12} \times \sqrt[3]{\frac{18}{64}} = \frac{1}{12} \times \frac{1}{4} \times \sqrt[3]{18} =$ $\frac{1}{48}\sqrt[3]{18}$, the product required.

3. Required the product of $3\sqrt{2}$ and $2\sqrt{8}$. Ans. 24.

4. Required the product of $\frac{1}{3}\sqrt[3]{4}$ and $\frac{3}{4}\sqrt[3]{12}$. Ans. $\frac{1}{2}\sqrt[3]{6}$.

5. To find the product of $\frac{5}{3}\sqrt{\frac{3}{8}}$ and $\frac{9}{10}\sqrt{\frac{2}{5}}$. Ans. $\frac{3}{20}\sqrt{15}$.

6. Required the product of $2\sqrt[3]{14}$ and $3\sqrt[3]{4}$. Ans. $12\sqrt[3]{7}$.

7. Required the product of $2a^{\frac{2}{3}}$ and $a^{\frac{4}{3}}$. Ans. $2a^2$.

8. Required the product of $(a+b)^{\frac{1}{2}}$ and $(a+b)^{\frac{3}{4}}$.

9. Required the product of $2x + \sqrt{b}$ and $2x - \sqrt{b}$.

10. Required the product of $(a + 2\sqrt{b})^{\frac{1}{2}}$, and $(a - 2\sqrt{b})^{\frac{1}{2}}$.

11. Required the product of $2x^{\frac{1}{n}}$ and $3x^{\frac{1}{m}}$.

12. Required the product of $4x^{\frac{1}{}}$ and $2y^{\frac{1}{n}}$.

PROBLEM VII.

To Divide one Surd Quantity by another.

REDUCE the surds to the same index, if necessary; then take the quotient of the rational quantities, and annex it to the quotient of the surds, and it will give the whole quotient required; which may be reduced to more simple terms, if requisite.

EXAMPLES.

1. Required to divide $6\sqrt{96}$ by $3\sqrt{8}$.

Here $6 \div 3 \cdot \sqrt{(96 \div 8)} = 2\sqrt{12} = 2\sqrt{(4 \times 3)} = 2 \times 2\sqrt{3}$ $= 4\sqrt{3}$, the quotient required.

2. Required to divide $12\sqrt[3]{280}$ by $3\sqrt[3]{5}$.

Here $12 \div 3 = 4$, and $280 \div 5 = 56 = 8 \times 7 = 2^3.7$;
Therefore $4 \times 2 \times \sqrt[3]{7} = 8\sqrt[3]{7}$, is the quotient required.

3. Let

3. Let $4\sqrt{50}$ be divided by $2\sqrt{5}$. Ans. $2\sqrt{10}$.

4. Let $6\sqrt[3]{100}$ be divided by $3\sqrt[3]{5}$. Ans. $2\sqrt[3]{20}$.

5. Let $\frac{5}{6}\sqrt{\frac{1}{50}}$ be divided by $\frac{3}{4}\sqrt{\frac{2}{5}}$. Ans. $\frac{1}{18}\sqrt{5}$.

6. Let $\frac{1}{4}\sqrt[3]{\frac{3}{16}}$ be divided by $\frac{3}{5}\sqrt[3]{\frac{2}{5}}$. Ans. $\frac{5}{16}\sqrt[3]{30}$.

7. Let $\frac{4}{5}\sqrt{a}$, or $\frac{4}{5}a^{\frac{1}{2}}$, be divided by $\frac{2}{3}a^{\frac{1}{3}}$. Ans. $\frac{6}{5}a^{\frac{1}{6}}$.

8. Let $a^{\frac{4}{3}}$ be divided by $a^{\frac{2}{3}}$.

9. To divide $3a^{\frac{1}{n}}$ by $4a^{\frac{1}{m}}$.

PROBLEM VIII.

To Involve or Raise Surd Quantities to any Power.

RAISE both the rational part and the surd part. Or multiply the index of the quantity by the index of the power to which it is to be raised, and to the result annex the power of the rational parts, which will give the power required.

EXAMPLES.

1. Required to find the square of $\frac{3}{4}a^{\frac{1}{2}}$.

First, $(\frac{3}{4})^2=\frac{3}{4}\times\frac{3}{4}=\frac{9}{16}$, and $(a^{\frac{1}{2}})^2=a^{\frac{1}{2}\times 2}=a^{\frac{2}{2}}=a$.
Therefore $(\frac{3}{4}a^{\frac{1}{2}})^2=\frac{9}{16}a$, is the square required.

2. Required to find the square of $\frac{1}{2}a^{\frac{2}{3}}$.

First, $\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$, and $(a^{\frac{2}{3}})^2=a^{\frac{4}{3}}=a\sqrt[3]{a}$;
Therefore $(\frac{1}{2}a^{\frac{2}{3}})^3=\frac{1}{4}a\sqrt[3]{a}$ is the square required.

3. Required to find the cube of $\frac{2}{3}\sqrt{6}$ or $\frac{2}{3}\times 6^{\frac{1}{2}}$.

First, $(\frac{2}{3})^3=\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}=\frac{8}{27}$, and $(6^{\frac{1}{2}})^3=6^{\frac{3}{2}}=6\sqrt{6}$;
Theref. $(\frac{2}{3}\sqrt{6})^3=\frac{8}{27}\times 6\sqrt{6}=\frac{16}{9}\sqrt{6}$, the cube required.

4. Required the square of $2\sqrt[3]{2}$. Ans. $4\sqrt[3]{4}$.

5. Required the cube of $3^{\frac{1}{2}}$, or $\sqrt{3}$. Ans. $3\sqrt{3}$.

6. Required the 3d power of $\frac{1}{3}\sqrt{3}$. Ans. $\frac{1}{9}\sqrt{3}$.

7. Required to find the 4th power of $\frac{1}{2}\sqrt{2}$. Ans. $\frac{1}{4}$.

8. Required

8. Required to find the *m*th power of $a^{\frac{1}{n}}$.

9. Required to find the square of $2 + \sqrt{3}$.

PROBLEM IX.

To Evolve or Extract the Roots of Surd Quantities.*

EXTRACT both the rational part and the surd part. Or divide the index of the given quantity by the index of the root to be extracted; then to the result annex the root of the rational part, which will give the root required.

EXAMPLES.

1. Required to find the square root of $16\sqrt{6}$.

First, $\sqrt{16} = 4$, and $(6^{\frac{1}{2}})^{\frac{1}{2}} = 6^{\frac{1}{2} \div 2} = 6^{\frac{1}{4}}$;

theref. $(16\sqrt{6})^{\frac{1}{2}} = 4 . 6^{\frac{1}{4}} = 4\sqrt[4]{6}$, is the sq. root required.

2. Required to find the cube root of $\frac{1}{27}\sqrt{3}$.

First, $\sqrt[3]{\frac{1}{27}} = \frac{1}{3}$, and $(\sqrt{3})^{\frac{1}{3}} = 3^{\frac{1}{2} \div 3} = 3^{\frac{1}{6}}$;

theref. $(\frac{1}{27}\sqrt{3})^{\frac{1}{3}} = \frac{1}{3} . 3^{\frac{1}{6}} = \frac{1}{3}\sqrt[6]{3}$, is the cube root required.

3. Required the square root of 6^3. Ans. $6\sqrt{6}$.

4. Required the cube root of $\frac{1}{8}a^3b$. Ans. $\frac{1}{2}a\sqrt[3]{6}$.

5. Required the 4th root of $16a^2$. Ans. $2\sqrt{a}$.

6. Required to find the *m*th root of $x^{\frac{1}{n}}$.

7. Required the square root of $a^2 - 6a\sqrt{b} + 9b$.

---

* The square root of a binomial or residual surd, $a + b$, or $a - b$, may be found thus: Take $\sqrt{a^2 - b^2} = c$;

then $\sqrt{a + b} = \sqrt{\frac{a + c}{2}} + \sqrt{\frac{a - c}{2}}$;

and $\sqrt{a - b} = \sqrt{\frac{a + c}{2}} - \sqrt{\frac{a - c}{2}}$.

Thus, the square root of $4 + 2\sqrt{3} = 1 + \sqrt{3}$;
and the square root of $6 - 2\sqrt{5} = \sqrt{5} - 1$.
But for the cube, or any higher root, no general rule is known.

INFINITE SERIES.

An Infinite Series is formed either from division, dividing by a compound divisor, or by extracting the root of a compound surd quantity; and is such as, being continued, would run on infinitely, in the manner of a continued decimal fraction.

But, by obtaining a few of the first terms, the law of the progression will be manifest; so that the series may thence be continued, without actually performing the whole operation.

PROBLEM I.

To Reduce Fractional Quantities into Infinite Series by Division.

Divide the numerator by the denominator, as in common division; then the operation, continued as far as may be thought necessary, will give the infinite series required.

EXAMPLES.

1. To change $\frac{2ab}{a+b}$ into an infinite series.

$$a+b)\,2ab\,..\,(2b-\frac{2b^2}{a}+\frac{2b^3}{a^2}-\frac{2b^4}{a^3}+\&c.$$

$$\begin{array}{l}
2ab+2b^2 \\
\hline
\quad -2b^2 \\
\quad -2b^2-\frac{2b^3}{a} \\
\hline
\qquad \frac{2b^3}{a} \\
\qquad \frac{2b^3}{a}+\frac{2b^4}{a^2} \\
\hline
\qquad\quad -\frac{2b^4}{a^2} \\
\qquad\quad -\frac{2b^4}{a^2}-\frac{2b^5}{a^3} \\
\hline
\qquad\qquad \frac{2b^5}{a^3},\ \&c.
\end{array}$$

2. Let

2\. Let $\dfrac{1}{1-a}$ be changed into an infinite series.

$$\begin{array}{l} 1-a)\ 1 \ldots\ldots (1+a+a^2+a^3+a^4+\&c. \\ \quad\quad 1-a \\ \hline \quad\quad\quad a \\ \quad\quad\quad a-a^2 \\ \hline \quad\quad\quad\quad a^2 \\ \quad\quad\quad\quad a^2-a^3 \\ \hline \quad\quad\quad\quad\quad a^3 \\ \quad\quad\quad\quad\quad a^3-a^4 \\ \hline \quad\quad\quad\quad\quad\quad a^4 \end{array}$$

3\. Expand $\dfrac{b}{a+c}$ into an infinite series.

Ans. $\dfrac{b}{a} \times (1 - \dfrac{c}{a} + \dfrac{c^2}{a^2} - \dfrac{c^3}{a^3} + \&c.)$

4\. Expand $\dfrac{a}{a-b}$ into an infinite series.

Ans. $1 + \dfrac{b}{a} + \dfrac{b^2}{a^2} + \dfrac{b^3}{a^3} + \&c.$

5\. Expand $\dfrac{1-x}{1+x}$ into an infinite series.

Ans. $1 - 2x + 2x^2 - 2x^3 + 2x^4$, &c.

6\. Expand $\dfrac{a^2}{(a+b)^2}$ into an infinite series.

Ans. $1 - \dfrac{2b}{a} + \dfrac{3b^2}{a^2} - \dfrac{4b^3}{a^3}$, &c.

7\. Expand $\dfrac{1}{1+1} = \frac{1}{2}$, into an infinite series.

PROBLEM II.

To Reduce a Compound Surd into an Infinite Series.

Extract the root as in common arithmetic; then the operation, continued as far as may be thought necessary, will give the series required. But this method is chiefly of use in extracting the square root, the operation being too tedious for the higher powers.

EXAMPLES.

EXAMPLES.

1. **Extract the root of a^2-x^2 in an infinite series.**

$$a^2-x^2\ \left(a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}\ \&c.\right.$$

$$a^2$$

$$2a-\frac{x^2}{2a}\Big)\ -x^2$$

$$-x^2+\frac{x^4}{4a^2}$$

$$2a-\frac{x^2}{a}-\frac{x^4}{8a^3}\Big)-\frac{x^4}{4a^2}$$

$$-\frac{x^4}{4a^2}+\frac{x^6}{8a^4}+\frac{x^8}{64a^6}$$

$$2a-\frac{x^2}{a}-\frac{x^4}{4a^3}\ \&c.\Big)\ -\frac{x^6}{8a^4}-\frac{x^8}{64a^6}$$

$$-\frac{x^6}{8a^4}+\frac{x^8}{16a^6}\ \&c.$$

$$-\frac{5x^8}{64a^6}\ \&c.$$

2. Expand $\sqrt{1+1}=\sqrt{2}$, into an infinite series.
Ans. $1+\frac{1}{2}-\frac{1}{8}+\frac{1}{16}-\frac{5}{128}$ &c.

3. Expand $\sqrt{1-1}$ into an infinite series.
Ans. $1-\frac{1}{2}-\frac{1}{8}-\frac{1}{16}-\frac{5}{128}$ &c.

4. Expand $\sqrt{a^2+x}$ into an infinite series.

5. Expand $\sqrt{a^2-2bx-x^2}$ to an infinite series.

PROBLEM III.

To Extract any Root of a Binomial: or to Reduce a Binomial Surd into an Infinite Series.

This will be done by substituting the particular letters of the binomial, with their proper signs, in the following general theorem or formula, viz.

$$(\mathrm{P}+\mathrm{PQ})^{\frac{m}{n}}=\mathrm{P}^{\frac{m}{n}}+\frac{m}{n}\mathrm{AQ}+\frac{m-n}{2n}\mathrm{BQ}+\frac{m-2n}{3n}\mathrm{CQ}+\ \&c.$$

and it will give the root required: observing that P denotes the first term, Q the second term divided by the first, $\frac{m}{n}$ the index of the power or root; and A, B, C, D, &c, denote the several foregoing terms with their proper signs.

EXAMPLES.

1. To extract the sq. root of a^2+b^2, in an infinite series.

Here $P = a^2$, $Q = \frac{b^2}{a^2}$, and $\frac{m}{n} = \frac{1}{2}$: therefore

$P^{\frac{m}{n}} = (a^2)^{\frac{m}{n}} = (a^2)^{\frac{1}{2}} = a = A$, the 1st term of the series.

$\frac{m}{n}AQ = \frac{1}{2} \times a \times \frac{b^2}{a^2} = \frac{b^2}{2a} = B$, the 2d term.

$\frac{m-n}{2n}BQ \times \frac{1-2}{4} \times \frac{b^2}{2a} \times \frac{b^2}{a^2} = -\frac{b^4}{2.4a^3} = C$, the 3d term

$\frac{m-2n}{3n}CQ = \frac{1-4}{6} \times -\frac{b^4}{2.4a^3} \times \frac{b^2}{a^2} = \frac{3b^6}{2.4.6a^5} = D$ the 4th.

Hence $a + \frac{b^2}{2a} - \frac{b^4}{2.4a^3} + \frac{3.b^6}{2.4.6a^5} -$ &c, or

$a + \frac{b^2}{2a} - \frac{b^4}{8a^3} + \frac{b^6}{16a^5} - \frac{5b^8}{128a^7}$ &c, is the series required.

2. To find the value of $\frac{1}{(a-x)^2}$, or its equal $(a-x)^{-2}$, in an infinite series*.

---

* *Note.* To facilitate the application of the rule to fractional examples, it is proper to observe, that any surd may be taken from the denominator of a fraction and placed in the numerator, and vice versa, by only changing the sign of its index. Thus,

$\frac{1}{x^2} = 1 \times x^{-2}$ or only x^{-2}; and $\frac{1}{(a+b)^2} = 1 \times (a+b)^{-2}$ or $(a+b)^{-2}$; and $\frac{a^2}{(a+x)^2} = a^2\,(a+x)^{-2}$; and $\frac{x^{\frac{1}{2}}}{x^{\frac{1}{3}}} = x^{\frac{1}{2}} \times x^{\frac{-1}{3}}$; also $\frac{(a^2+x^2)^{\frac{1}{2}}}{(a^2-x^2)^{\frac{1}{2}}} = (a^2+x^2)^{\frac{1}{2}} \times (a^2-x^2)^{\frac{-1}{2}}$; &c.

Here

Here $P = a$, $Q = \frac{-x}{a} = -a^{-1}x$, and $\frac{m}{n} = \frac{-2}{1} = -2$; theref.

$P^{\frac{m}{n}} = (a)^{-2} = a^{-2} = \frac{1}{a^2} = A$, the 1st term of the series.

$\frac{m}{n}AQ = -2 \times \frac{1}{a^2} \times \frac{-x}{a} = \frac{2x}{a^3} = 2a^{-3}x = B$, the 2d term.

$\frac{m-n}{2n}BQ = -\frac{3}{2} \times \frac{2x}{a^3} \times \frac{-x}{a} = \frac{3x^2}{a^4} = 3a^{-4}x^2 = C$, the 3d.

$\frac{m-2n}{3n}CQ = -\frac{4}{3} \times \frac{3x^2}{a^4} \times \frac{-x}{a} = \frac{4x^3}{a^5} = 4a^{-5}x^3 = D$.

Hence $a^{-2} + 2a^{-3}x + 3a^{-4}x^2 + 4a^{-5}x^3 +$ &c, or

$\frac{1}{a^2} + \frac{2x}{a^3} + \frac{3x^2}{a^4} + \frac{4x^3}{a^5} + \frac{5x^4}{a^6}$ &c, is the series required.

3. To find the value of $\frac{a^2}{a-x}$, in an infinite series.

Ans. $a + x + \frac{x^2}{a} + \frac{x^3}{a^2} + \frac{x^4}{a^3}$ &c.

4. To expand $\sqrt{\frac{1}{(a^2+x^2)}}$ or $\frac{1}{(a^2+x^2)^{\frac{1}{2}}}$ in a series.

Ans. $\frac{1}{a} - \frac{x^2}{2a^3} + \frac{3x^4}{8a^5} - \frac{5x^6}{16a^7}$ &c.

5. To expand $\frac{a^2}{(a-b)^2}$ in an infinite series.

Ans. $1 + \frac{2b}{a} + \frac{3b^2}{a^2} + \frac{4b^3}{a^3} + \frac{5b^4}{a^4}$ &c.

6. To expand $\sqrt{a^2-x^2}$ or $(a^2-x^2)^{\frac{1}{2}}$ in a series.

Ans. $a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}$ &c.

7. Find the value of $\sqrt[3]{(a^3-b^3)}$ or $(a^3-b^3)^{\frac{1}{3}}$ in a series.

Ans. $a - \frac{b^3}{3a^2} - \frac{b^6}{9a^5} - \frac{5b^9}{81a^8}$ &c.

8. To find the value of $\sqrt[5]{(a^5+x^5)}$ or $(a^5+x^5)^{\frac{1}{5}}$ in a series.

Ans. $a + \frac{x^5}{5a^4} - \frac{2x^{10}}{25a^9} + \frac{6x^{15}}{125a^{14}}$ &c.

9. To

9. To find the square root of $\frac{a^2 - b^2}{a^2 + b^2}$ in an infinite series.

$$\text{Ans. } 1 - \frac{b}{a} + \frac{x^2}{2a^2} - \frac{x^3}{2a^3} \text{ \&c.}$$

10. Find the cube root of $\frac{a^3}{a^3 + b^3}$ in a series.

$$\text{Ans. } 1 - \frac{b^3}{3a^3} + \frac{3b^6}{9a^6} - \frac{14b^9}{81a^9} \text{ \&c.}$$

ARITHMETICAL PROPORTION.

ARITHMETICAL PROPORTION is the relation between two numbers with respect to their difference.

Four quantities are in Arithmetical Proportion, when the difference between the first and second is equal to the difference between the third and fourth. Thus, 4, 6, 7, 9, and a, $a + d$, b, $b + d$, are in arithmetical proportion.

Arithmetical Progression is when a series of quantities have all the same common difference, or when they either increase or decrease by the same common difference. Thus, 2, 4, 6, 8, 10, 12, &c, are in arithmetical progression, having the common difference 2; and a, $a + d$, $a + 2d$, $a + 3d$, $a + 4d$, $a + 5d$, &c, are series in arithmetical progression, the eommon difference being d.

The most useful part of arithmetical proportion is contained in the following theorems:

1. When four quantities are in Arithmetical Proportion, the sum of the two extremes is equal to the sum of the two means. Thus, in the arithmeticals 4, 6, 7, 9, the sum $4 + 9 = 6 + 7 = 13$: and in the arithmeticals a, $a + d$, b, $b + d$, the sum $a + b + d = a + b + d$.

2. In any continued arithmetical progression, the sum of the two extremes is equal to the sum of any two terms at an equal distance from them.

Thus,

Thus, if the series be 1, 3, 5, 7, 9, 11, &c.
Then $1 + 11 = 3 + 9 = 5 + 7 = 12$.

3. The last term of any increasing arithmetical series, is equal to the first term increased by the product of the common difference multiplied by the number of terms less one; but in a decreasing series, the last term is equal to the first term lessened by the said product.

Thus, the 20th term of the series 1, 3, 5, 7, 9, &c, is $= 1 + 2\,(20-1) = 1 + 2 \times 19 = 1 + 38 = 39$.

And the nth term of a, $a-d$, $a-2d$, $a-3d$, $a-4d$, &c, is $= a - (n-1) \times d = a - (n-1)d$.

4. The sum of all the terms in any series in arithmetical progression, is equal to half the sum of the two extremes multiplied by the number of terms.

Thus, the sum of 1, 3, 5, 7, 9, &c, continued to the 10th term, is $= \frac{(1 + 19) \times 10}{2} = \frac{20 \times 10}{2} = 10 \times 10 = 100$.

And the sum of n terms of a, $a+d$, $a+2d$, $a+3d$, to $a+md$, is $= (a+a+md) \cdot \frac{n}{2} = (a + \frac{1}{2}md)\,n$.

EXAMPLES FOR PRACTICE.

1. The first term of an increasing arithmetical series is 1, the common difference 2, and the number of terms 21; required the sum of the series?

First, $1 + 2 \times 20 = 1 + 40 = 41$, is the last term.

Then $\frac{1 + 41}{2} \times 20 = 21 \times 20 = 420$, the sum required.

2. The first term of a decreasing arithmetical series is 199, the common difference 3, and the number of terms 37; required the sum of the series?

First, $199 - 3 \cdot 66 = 199 - 198 = 1$, is the last term.

Then $\frac{199 + 1}{2} \times 67 = 100 \times 67 = 6700$, the sum required.

3. To find the sum of 100 terms of the natural numbers 1, 2, 3, 4, 5, 6, &c. Ans. 5050.

9. To find the square root of $\frac{a^2 - b^2}{a^2 + b^2}$ in a ... numbers

Ans. 9811.

Ans. $1 - \frac{b}{a} +$ ... is 10, ... terms 21; Ans. 140.

10. Find the cube root of $\frac{a^3}{a^3 + b^3}$ in a ... round, in a ... each other; ... one by ... the first ... yards.

Ans. $1 - \frac{b^3}{3a^3} +$

ARITHMETICAL PRO... PROGRESSION ... AIRS.

Arithmetical Proportio... two numbers with respect to their ...

Four quantities are in Arithm... difference between the first an... ference between the third an... and a, $a + d$, b, $b + d$, are in ari...

... thirty ranks, ... one man only, the ... second

Arithmetical Progression is ... have all the same common dif... increase or decrease by the sa... 2, 4, 6, 8, 10, 12, &c, are in ... ing the common difference 2; ... $a + 4d$, $a + 5d$, &c. are seri... the common difference bein...

... of the arithmetical ... is equal to the square (n^2) ... there will ... 1, 2, 3, &c, terms. ... of 1 term, ... terms, ... terms, ... terms, &c.

The most useful part ... tained in the following th... $= 1 + 2n - 2 = 2n - 1$ ... to this last term

1. When four quanti... the sum of the two extr... means. Thus, in the ari... $9 = 6 + 7 = 13$; and in ... the sum $a + b + d = a +$ ...

... the extremes, or ... theorem, $n \times n$ ... appears in general, ... same as the num- ... terms, is the same

2. In any continued ... the two extremes is equ... equal distance from

... the Arithmetic, ... a body of troops ... ranks exceed each other.

second of 3, the third of 5; and so on: What is the strength of such a triangular battalion?

Answer, 900 men.

QUESTION II.

A detachment having 12 successive days to march, with orders to advance the first day only 2 leagues, the second $3\frac{1}{2}$, and so on, increasing $1\frac{1}{2}$ league each day's march: What is the length of the whole march, and what is the last day's march?

Answer, the last day's march is $18\frac{1}{2}$ leagues, and 123 leagues is the length of the whole march.

QUESTION III.

A brigade of sappers *, having carried on 15 yards of sap the first night, the second only 13 yards, and so on, decreasing 2 yards every night, till at last they carried on in one night only 3 yards: What is the number of nights they were employed; and what is the whole length of the sap?

Answer, they were employed 7 nights, and the length of the whole sap was 63 yards.

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other by an equal number of men: if the first rank consist of one man only, and the difference between the ranks be also 1, then its form is that of an equilateral triangle; and when the difference between the ranks is more than 1, its form may then be an isosceles or scalene triangle. The practice of forming troops in this order, which is now laid aside, was formerly held in greater esteem than forming them in a solid square, as admitting of a greater front, especially when the troops were to make simply a stand on all sides.

* A brigade of sappers, consists generally of 8 men, divided equally into two parties. While one of these parties is advancing the sap, the other is furnishing the gabions, fascines, and other necessary implements: and when the first party is tired, the second takes its place, and so on, till each man in turn has been at the head of the sap. A sap is a small ditch, between 3 and 4 feet in breadth and depth; and is distinguished from the trench by its breadth only, the trench having between 10 and 15 feet breadth. As an encouragement to sappers, the pay for all the work carried on by the whole brigade, is given to the survivors.

QUESTION IV.

A number of gabions* being given to be placed in six ranks, one above the other, in such a manner as that each rank exceeding one another equally, the first may consist of 4 gabions, and the last of 9: What is the number of gabions in the six ranks; and what is the difference between each rank?

Answer, the difference between the ranks will be 1, and the number of gabions in the six ranks will be 39.

QUESTION V.

Two detachments, distant from each other 37 leagues, and both designing to occupy an advantageous post equi-distant from each other's camp, set out at different times; the first detachment increasing every day's march 1 league and a half, and the second detachment increasing each day's march 2 leagues: both the detachments arrive at the same time; the first after 5 days' march, and the second after 4 days' march: What is the number of leagues marched by each detachment each day?

The progression $\frac{7}{10}$, $2\frac{2}{10}$, $3\frac{7}{10}$, $5\frac{2}{10}$, $6\frac{7}{10}$, answers the conditions of the first detachment: and the progression $1\frac{5}{8}$, $3\frac{5}{8}$, $5\frac{5}{8}$, $7\frac{5}{8}$, answers the conditions of the second detachment.

QUESTION VI.

A deserter, in his flight, travelling at the rate of 8 leagues a day; and a detachment of dragoons being sent after him, with orders to march the first day only 2 leagues, the second 5 leagues, the third 8 leagues, and so on: What is the number of days necessary for the detachment to overtake the deserter, and what will be the number of leagues marched before he is overtaken?

Answer, 5 days are necessary to overtake him; and consequently 40 leagues will be the extent of the march.

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* Gabions are baskets, open at both ends, made of ozier twigs, and of a cylindrical form: those made use of at the trenches are 2 feet wide, and about 3 feet high; which, being filled with earth, serve as a shelter from the enemy's fire: and those made use of to construct batteries, are generally higher and broader. There is another sort of gabion, made use of to raise a low parapet: its height is from 1 to 2 feet, and 1 foot wide at top, but somewhat less at bottom, to give room for placing the muzzle of a firelock between them: these gabions serve instead of sand bags. A sand bag is generally made to contain about a cubical foot of earth.

QUESTION VII.

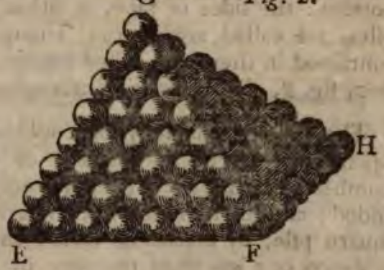
A convoy* distant 35 leagues, having orders to join its camp, and to march at the rate of 5 leagues per day; its escort departing at the same time, with orders to march the first day only half a league, and the last day $9\frac{1}{2}$ leagues; and both the escort and convoy arriving at the same time: At what distance is the escort from the convoy at the end of each march?

OF COMPUTING SHOT OR SHELLS IN A FINISHED PILE.

SHOT and Shells are generally piled in three different forms, called triangular, square, or oblong piles, according as their base is either a triangle, a square, or a rectangle.

Fig. 1.

Fig. 2.



ABCD, fig. 1, is a triangular pile,
EFGH, fig. 2, is a square pile.

Fig.

ABCDEF, fig. 3, is an oblong pile.

* By convoy is generally meant a supply of ammunition or provisions, conveyed to a town or army. The body of men that *guard this supply, is called* escort.

A triangular pile is formed by the continual laying of triangular horizontal courses of shot one above another, in such a manner, as that the sides of these courses, called rows, decrease by unity from the bottom row to the top row, which ends always in 1 shot.

A square pile is formed by the continual laying of square horizontal courses of shot one above another, in such a manner, as that the sides of these courses decrease by unity from the bottom to the top row, which ends also in 1 shot.

In the triangular and the square piles, the sides or faces being equilateral triangles, the shot contained in those faces form an arithmetical progression, having for first term unity, and for last term and number of terms, the shot contained in the bottom row; for the number of horizontal rows, or the number counted on one of the angles from the bottom to the top, is always equal to those counted on one side in the bottom: the sides or faces in either the triangular or square piles, are called arithmetical triangles; and the numbers contained in these, are called triangular numbers: ABC, fig. 1, EFG, fig. 2, are arithmetical triangles.

The oblong pile may be conceived as formed from the square pile ABCD; to one side or face of which, as AD, a number of arithmetical triangles equal to the face have been added: and the number of arithmetical triangles added to the square pile, by means of which the oblong pile is formed, is always one less than the shot in the top row; or, which is the same, equal to the difference between the bottom row of the greater side and that of the lesser.

QUESTION VIII.

To find the shot in the triangular pile ABCD, fig. 1, the bottom row AB consisting of 8 shot.

SOLUTION.

The proposed pile consisting of 8 horizontal courses, each of which forms an equilateral triangle; that is, the shot contained in these being in an arithmetical progression, of which the first and last term, as also the number of terms, are known; it follows, that the sum of these particular courses, or of the 8 progressions, will be the shot contained in the proposed pile; then

The

The shot of the first or lower triangular course will be	$\overline{8+1}\times 4 = 36$
the second - - - -	$\overline{7+1}\times 3\tfrac{1}{2} = 28$
the third - - - -	$\overline{6+1}\times 3 = 21$
the fourth - - - -	$\overline{5+1}\times 2\tfrac{1}{2} = 15$
the fifth - - - -	$\overline{4+1}\times 2 = 10$
the sixth - - - -	$\overline{3+1}\times 1\tfrac{1}{2} = 6$
the seventh - - - -	$\overline{2+1}\times 1 = 3$
the eighth - - - -	$\overline{1+1}\times \tfrac{1}{2} = 1$
	Total - 120 shot in the pile proposed.

QUESTION IX.

To find the shot of the square pile EFGH, fig. 2, the bottom row EF consisting of 8 shot.

SOLUTION.

The bottom row containing 8 shot, and the second only 7; that is, the rows forming the progression, 8, 7, 6, 5, 4, 3, 2, 1, in which each of the terms being the square root of the shot contained in each separate square course employed in forming the square pile; it follows, that the sum of the squares of these roots will be the shot required; and the sum of the squares divided by 8, 7, 6, 5, 4, 3, 2, 1, being 204, expresses the shot in the proposed pile.

QUESTION X.

To find the shot of the oblong pile ABCDEF, fig. 3; in which BF = 16, and BC = 7.

SOLUTION.

The oblong pile proposed, consisting of the square pile ABCD, whose bottom row is 7 shot; besides 9 arithmetical triangles or progressions, in which the first and last term, as also the number of terms, are known; it follows, that,

if to the contents of the square pile	-	140
we add the sum of the 9th progression	-	252
their total gives the contents required	-	392 shot.

REMARK I.

The shot in the triangular and the square piles, as also the shot in each horizontal course, may at once be ascertained

tained by the following table: the vertical column A, contains the shot in the bottom row, from 1 to 20 inclusive; the column B contains the triangular numbers, or number of each course; the column C contains the sum of the triangular numbers, that is, the shot contained in a triangular pile, commonly called pyramidal numbers; the column D contains the square of the numbers of the column A, that is, the shot contained in each square horizontal course; and the column E contains the sum of these squares or shot in a square pile.

C	B	A	D	E
Pyramidal numbers.	Triangular numbers.	Natural numbers.	Square of the natural numbers.	Sumofthese square numbers.
1	1	1	1	1
4	3	2	4	5
10	6	3	9	14
20	10	4	16	30
35	15	5	25	55
56	21	6	36	91
84	28	7	49	140
120	36	8	64	204
165	45	9	81	285
220	55	10	100	385
286	66	11	121	506
364	78	12	144	650
455	91	13	169	819
560	105	14	196	1015
680	120	15	225	1240
816	136	16	256	1496
969	153	17	289	1785
1140	171	18	324	2109
1330	190	19	361	2470
1540	210	20	400	2870

Thus, the bottom row in a triangular pile, consisting of 9 shot, the contents will be 165; and when of 9 in the square pile, 285.—In the same manner, the contents either of a square or triangular pile being given, the shot in the bottom row may be easily ascertained.

The contents of any oblong pile by the preceding table may be also with little trouble ascertained, the less side not exceeding 20 shot, nor the difference between the less and *the greater* side 20. Thus, to find the shot in an oblong pile, the

the less side being 15, and the greater 35, we are first to find the contents of the square pile, by means of which the oblong pile may be conceived to be formed; that is, we are to find the contents of a square pile, whose bottom row is 15 shot; which being 1240, we are, secondly, to add these 1240 to the product 2400 of the triangular number 120, answering to 15, the number expressing the bottom row of the arithmetical triangle, multiplied by 20, the number of those triangles; and their sum, being 3640, expresses the number of shot in the proposed oblong pile.

REMARK II.

The following algebraical expressions, deduced from the investigations of the sums of the powers of numbers in arithmetical progression, which are seen upon many gunners' callipers *, serve to compute with ease and expedition the shot or shells in any pile.

That serving to compute any triangular pile, is represented by $\left.\right\} \dfrac{\overline{n+2}\times\overline{n+1}\times n}{6}$.

That serving to compute any square pile, is represented by $\left.\right\} \dfrac{\overline{n+1}\times\overline{2n+1}\times n}{6}$.

In each of these, the letter n represents the number in the bottom row: hence, in a triangular pile, the number in the bottom row being 30; then this pile will be $\overline{30+2}\times\overline{30+1}\times\frac{30}{6} = 4960$ shot or shells. In a square pile, the number in the bottom row being also 30; then this pile will be $\overline{30+1}\times\overline{60+1}\times\frac{30}{6} = 9455$ shot or shells.

That serving to compute any oblong pile, is represented by $\dfrac{\overline{2n+1+3m}\times\overline{n+1}\times n}{6}$, in which the letter n denotes

* Callipers are large compasses, with bowed shanks, serving to take the diameters of convex and concave bodies. The gunners' callipers consist of two thin rules or plates, which are moveable quite round a joint, by the plates folding one over the other: the length of each rule or plate is 6 inches, the breadth about 1 inch. It is usual to represent, on the plates, a variety of scales, tables, proportions, &c, such as are esteemed useful to be known by persons employed about artillery; but, except the measuring of the caliber of shot and cannon, and the measuring of saliant and re-entering angles, none of the articles, with which the callipers are usually filled, are essential to that instrument.

the

the number of courses, and the letter m the number of shot, less one, in the top row: hence, in an oblong pile the number of courses being 30, and the top row 31; this pile will be $\overline{60 + 1 + 90} \times \overline{30 + 1} \times \frac{30}{6} = 23405$ shot or shells.

GEOMETRICAL PROPORTION.

Geometrical Proportion contemplates the relation of quantities considered as to what part or what multiple one is of another, or how often one contains, or is contained in, another.—Of two quantities compared together, the first is called the Antecedent, and the second the Consequent. Their ratio is the quotient which arises from dividing the one by the other.

Four Quantities are proportional, when the two couplets have equal ratios, or when the first is the same part or multiple of the second, as the third is of the fourth. Thus, 3, 6, 4, 8, and a, ar, b, br, are geometrical proportionals. For $\frac{6}{3} = \frac{8}{4} = 2$, and $\frac{ar}{a} = \frac{br}{b} = r$. And they are stated thus, $3 : 6 :: 4 : 8$, &c.

Direct Proportion is when the same relation subsists between the first term and the second, as between the third and the fourth: As in the terms above. But Reciprocal, or Inverse Proportion, is when one quantity increases in the same proportion as another diminishes: As in these, 3, 6, 8, 4; and these, a, ar, br, b.

The Quantities are in geometrical progression, or continuous proportion, when every two terms have always the same ratio, or when the first has the same ratio to the second as the second to the third, and the third to the fourth, &c. Thus, 2, 4, 8, 16, 32, 64, &c, and a, ar, ar^2, ar^3, ar^4, ar^5, &c, are series in geometrical progression.

The most useful part of geometrical proportion is contained in the following theorems; which are similar to those in Arithmetical Proportion, using multiplication for addition, &c.

1. When

1. When four quantities are in geometrical proportion, the product of the two extremes is equal to the product of the two means. As in these, 3, 6, 4, 8, where $3 \times 8 = 6 \times 4 = 24$; and in these, a, ar, b, br, where $a \times br = ar \times b = abr$.

2. When four quantities are in geometrical proportion, the product of the means divided by either of the extremes gives the other extreme. Thus, if $3 : 6 :: 4 : 8$, then $\frac{6 \times 4}{3} = 8$, and $\frac{6 \times 4}{8} = 3$; also if $a : ar :: b : br$, then $\frac{abr}{a} = br$, or $\frac{abr}{br} = a$. And this is the foundation of the Rule of Three.

3. In any continued geometrical progression, the product of the two extremes, and that of any other two terms, equally distant from them, are equal to each other, or equal to the square of the middle term when there is an odd number of them. So, in the series 1, 2, 4, 8, 16, 32, 64, &c, it is $1 \times 64 = 2 \times 32 = 4 \times 16 = 8 \times 8 = 64$.

4. In any continued geometrical series, the last term is equal to the first multiplied by such a power of the ratio as is denoted by 1 less than the number of terms. Thus, in the series, 3, 6, 12, 24, 48, 96, &c, it is $3 \times 2^5 = 96$.

5. The sum of any series in geometrical progression, is found by multiplying the last term by the ratio, and dividing the difference of this product and the first term by the difference between 1 and the ratio. Thus, the sum of 3, 6, 12, 24, 48, 96, 192, is $\frac{192 \times 2 - 3}{2 - 1} = 384 - 3 = 381$. And the sum of n terms of the series a, ar, ar^2, ar^3, ar^4, &c, to ar^{n-1}, is $\frac{ar^{n-1} \times r - a}{r - 1} = \frac{ar^n - a}{r - 1} = \frac{r^n - 1}{r - 1} a$.

6. When four quantities, a, ar, b, br, or 2, 6, 4, 12, are proportional; then any of the following forms of those quantities are also proportional, viz.

1. Directly $a : ar :: b : br$; or $2 : 6 :: 4 : 12$.
2. Inversely, $ar : a :: br : b$; or $6 : 2 :: 12 : 4$.
3. Alternately, $a : b :: ar : br$; or $2 : 4 :: 6 : 12$.

4. Com-

4. Compoundedly, $a : a + ar :: b : b + br$; or $2 : 8 :: 4 : 16$.

5. Dividedly, $a : ar - a :: b : br - b$; or $2 : 4 :: 4 : 8$.

6. Mixed, $ar + a : ar - a :: br + b : br - b$; or $8 : 4 :: 16 : 8$.

7. Multiplication, $ac : arc :: bc : brc$; or $2.3 : 6.3 :: 4 : 12$.

8. Division, $\frac{a}{c} : \frac{ar}{c} :: b : br$; or $1 : 3 :: 4 : 12$.

9. The numbers $a, b, c, d,$ are in harmonical proportion, when $a : d :: a \backsim b : c \backsim d$; or when their reciprocals $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d},$ are in arithmetical proportion.

EXAMPLES.

1. Given the first term of a geometrical series 1, the ratio 2, and the number of terms 12; to find the sum of the series?

First, $1 \times 2^{11} = 1 \times 2048$, is the last term.

Then $\frac{2048 \times 2 - 1}{2 - 1} = \frac{4096 - 1}{1} = 4095$, the sum required.

2. Given the first term of a geometric series $\frac{1}{3}$, the ratio $\frac{1}{2}$, and the number of terms 8; to find the sum of the series?

First, $\frac{1}{3} \times (\frac{1}{2})^7 = \frac{1}{3} \times \frac{1}{128} = \frac{1}{384}$, is the last term.

Then $(\frac{1}{3} - \frac{1}{384} \times \frac{1}{2}) \div (1 - \frac{1}{2}) = (\frac{1}{3} - \frac{1}{768}) \div \frac{1}{2} = \frac{255}{768} \times \frac{2}{1}$
$= \frac{255}{384}$, the sum required.

3. Required the sum of 12 terms of the series 1, 3, 9, 27, 81, &c. Ans. 265720.

4. Required the sum of 12 terms of the series 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, $\frac{1}{81}$, &c. Ans. $\frac{265720}{177147}$.

5. Required the sum of 100 terms of the series 1, 2, 4, 8, 16, 32, &c. Ans. 1267650600228229401496703205375.

See more of Geometrical Proportion in the Arithmetic.

SIMPLE EQUATIONS.

An Equation is the expression of two equal quantities, with the sign of equality (=) placed between them. Thus, $10 - 4 = 6$ is an equation, denoting the equality of the quantities $10 - 4$ and 6.

Equations

Equations are either simple or compound. A Simple Equation, is that which contains only one power of the unknown quantity, without including different powers. Thus, $x-a=b+c$, or $ax^2=b$, is a simple equation, containing only one power of the unknown quantity x. But $x^2-2ax=b^2$ is a compound one.

GENERAL RULE.

Reduction of Equations, is the finding the value of the unknown quantity. And this consists in disengaging that quantity from the known ones; or in ordering the equation so, that the unknown letter or quantity may stand alone on one side of the equation, or of the mark of equality, without a co-efficient; and all the rest, or the known quantities, on the other side.—In general, the unknown quantity is disengaged from the known ones, by performing always the reverse operations. So, if the known quantities are connected with it by $+$ or addition, they must be subtracted; if by minus ($-$), or subtraction, they must be added; if by multiplication, we must divide by them; if by division, we must multiply; when it is in any power, we must extract the root; and when in any radical, we must raise it to the power. As in the following particular rules; which are founded on the general principle of performing equal operations on equal quantities; in which case it is evident that the results must still be equal, whether by equal additions, or subtractions, or multiplications, or divisions, or roots, or powers.

PARTICULAR RULE I.

WHEN known quantities are connected with the unknown by $+$ or $-$; transpose them to the other side of the equation, and change their signs. Which is only adding or subtracting the same quantities on both sides, in order to get all the unknown terms on one side of the equation, and all the known ones on the other side *.

Thus,

* Here it is earnestly recommended that the pupil be accustomed, at every line or step in the reduction of the equations, to name the particular operation to be performed on the equation in the last line, in order to produce the next form or state of the equation, in applying each of these rules, according as the particular form of the equation may require; applying them according to the order

Thus, if $x+5=8$; then transposing 5 gives $x=8-5=3$.

And, if $x-3+7=9$; then transposing the 3 and 7, gives $x=9+3-7=5$.

Also, if $x-a+b=cd$: then by transposing a and b, it is $x=a-b+cd$.

In like manner, if $5x-6=4x+10$, then by transposing 6 and $4x$, it is $5x-4x=10+6$, or $x=16$.

RULE II.

When the unknown term is multiplied by any quantity; divide all the terms of the equation by it.

Thus, if $ax=ab-4a$; then dividing by a, gives $x=b-4$.

And, if $3x+5=20$; then first transposing 5 gives $3x=15$; and then by dividing by 3, it is $x=5$.

In like manner, if $ax+3ab=4c^2$; then by dividing by a, it is $x+3b=\frac{4c^2}{a}$; and then transposing $3b$, gives $x=\frac{4c^2}{a}-3b$.

RULE III.

When the unknown term is divided by any quantity; we must then multiply all the terms of the equation by that divisor; which takes it away.

Thus, if $\frac{x}{4}=3+2$: then mult. by 4, gives $x=12+8=20$.

And, if $\frac{x}{a}=3b+2c-d$:

then by mult. a, it gives $x=3ab+2ac-ad$.

Also, if $\frac{3x}{5}-3=5+2$:

Then by transposing 3, it is $\frac{3}{5}x=10$.
And multiplying by 5, it is $3x=50$.
Lastly dividing by 3 gives $x=16\frac{2}{3}$.

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order in which they are here placed; and beginning every line with the words *Then by*, as in the following specimens of Examples; which two words will always bring to his recollection, that he is to pronounce what particular operation he is to perform on the last line, in order to give the next; allotting always a single line for each operation, and ranging the equations neatly just under each other, in the several lines, as they are successively *produced.*

RULE

RULE IV.

When the unknown quantity is included in any root or surd: transpose the rest of the terms, if there be any, by Rule 1; then raise each side to such a power as is denoted by the index of the surd; viz. square each side when it is the square root; cube each side when it is the cube root; &c. which clears that radical.

Thus, if $\sqrt{x} - 3 = 4$; then transposing 3, gives $\sqrt{x} = 7$;
And squaring both sides gives $x = 49$.

And, if $\sqrt{2x + 10} = 8$:
Then by squaring, it becomes $2x + 10 = 64$;
And by transposing 10, it is $2x = 54$;
Lastly, dividing by 2, gives $x = 27$.

Also, if $\sqrt[3]{3x+4} + 3 = 6$:
Then by transposing 3, it is $\sqrt[3]{3x+4} = 3$;
And by cubing, it is $3x + 4 = 27$;
Also, by transposing 4, it is $3x = 23$;
Lastly, dividing by 3, gives $x = 7\frac{2}{3}$.

RULE V.

When that side of the equation which contains the unknown quantity is a complete power, or can easily be reduced to one, by rule 1, 2, or 3: then extract the root of the said power on both sides of the equation; that is, extract the square root when it is a square power, or the cube root when it is a cube, &c.

Thus, if $x^2 + 8x + 16 = 36$, or $(x + 4)^2 = 36$:
Then by extracting the roots, it is $x + 4 = 6$;
And by transposing 4, it is $x = 6 - 4 = 2$.

And if $3x^2 - 19 = 21 + 35$.
Then, by transposing 19, it is $3x^2 = 75$;
And dividing by 3, gives $x^2 = 25$;
And extracting the root, gives $x = 5$.

Also, if $\frac{3}{4}x^2 - 6 = 24$.
Then transposing 6, gives $\frac{3}{4}x^2 = 30$;
And multiplying by 4, gives $3x^2 = 120$;
Then dividing by 3, gives $x^2 = 40$;
Lastly, extracting the root, gives $x = \sqrt{40} = 6{\cdot}324555$.

RULE

RULE VI.

WHEN there is any analogy or proportion, it is to be changed into an equation, by multiplying the two extreme terms together, and the two means together, and making the one product equal to the other.

Thus, if $2x : 9 :: 3 : 5$.
Then, mult. the extremes and means, gives $10x = 27$;
And dividing by 10, gives $x = 2\frac{7}{10}$.

And if $\frac{1}{4}x : a :: 5b : 2c$.
Then mult. extremes and means gives $\frac{1}{2}cx = 5ab$;
And multiplying by 2, gives $3cx = 10ab$;
Lastly, dividing by $3c$, gives $x = \frac{10ab}{3c}$.

Also, if $10 - x : \frac{2}{3}x :: 3 : 1$.
Then mult. extremes and means, gives $10 - x = 2x$;
And transposing x, gives $10 = 3x$;
Lastly, dividing by 3, gives $3\frac{1}{3} = x$.

RULE VII.

WHEN the same quantity is found on both sides of an equation, with the same sign, either plus or minus, it may be left out of both: and when every term in an equation is either multiplied or divided by the same quantity, it may be struck out of them all.

Thus, if $3x + 2a = 2a + b$:
Then, by taking away $2a$, it is $3x = b$.
And, dividing by 3, it is $x = \frac{1}{3}b$.

Also if there be $4ax + 6ab = 7ac$.
Then striking out or dividing by a, gives $4x + 6b = 7c$.
Then, by transposing $6b$, it becomes $4x = 7c - 6b$;
And then dividing by 4 gives $x = \frac{7}{4}c - \frac{3}{2}b$.

Again, if $\frac{2}{3}x - \frac{7}{3} = \frac{10}{3} - \frac{7}{3}$.
Then, taking away the $\frac{7}{3}$, it becomes $\frac{2}{3}x = \frac{10}{3}$;
And taking away the 3's, it is $2x = 10$;
Lastly, dividing by 2 gives $x = 5$.

MISCELLANEOUS EXAMPLES.

1. Given $7x - 18 = 4x + 6$; to find the value of x.
First, transposing 18 and $5x$ gives $3x = 24$;
Then dividing by 3, gives $x = 8$.

2. Given

2. Given $20-4x-12=92-10x$; to find x.
First transposing 20 and 12 and $10x$, gives $6x=84$;
Then dividing by 6, gives $x=14$.

3. Let $4ax-5b=3dx+2c$ be given; to find x.
First, by trans. $5b$ and $3dx$, it is $4ax-3dx=5b+2c$;
Then dividing by $4a-3d$, gives $x=\dfrac{5b+2c}{4a-3d}$.

4. Let $5x^2-12x=9x+2x^2$ be given; to find x.
First, by dividing by x, it is $5x-12=9+2x$;
Then transposing 12 and $2x$, gives $3x=21$;
Lastly, dividing by 3, gives $x=7$.

5. Given $9ax^3-15abx^2=6ax^3+12ax^2$; to find x.
First, dividing by $3ax^2$, gives $3x-5b=2x+4$;
Then transposing $5b$ and $2x$, gives $x=5b+4$.

6. Let $\dfrac{x}{3}-\dfrac{x}{4}+\dfrac{x}{5}=2$ be given, to find x.
First, multiplying by 3, gives $x-\frac{3}{4}x+\frac{3}{5}x=6$;
Then multiplying by 4, gives $x+\frac{12}{5}x=24$.
Also multiplying by 5, gives $17x=120$;
Lastly, dividing by 17, gives $x=7\frac{1}{17}$.

7. Given $\dfrac{x-5}{3}+\dfrac{x}{2}=12-\dfrac{x-10}{3}$; to find x.
First, mult. by 3, gives $x-5+\frac{3}{2}x=36-x+10$;
Then transposing 5 and x, gives $2x+\frac{3}{2}x=51$;
And multiplying by 2, gives $7x=102$;
Lastly, dividing by 7, gives $x=14\frac{4}{7}$.

8. Let $\sqrt{\dfrac{3x}{4}}+7=10$, be given; to find x.
First, transposing 7, gives $\sqrt{\frac{3}{4}x}=3$;
Then squaring the equation, gives $\frac{3}{4}x=9$;
Then dividing by 3, gives $\frac{1}{4}x=3$;
Lastly, multiplying by 4, gives $x=12$.

9. Let $2x+2\sqrt{a^2+x^2}=\dfrac{5a^2}{\sqrt{a^2+x^2}}$, be given; to find x.
First, mult. by $\sqrt{a^2+x^2}$, gives $2x\sqrt{a^2+x^2}+2a^2+2x^2=5a^2$.
Then transp. $2a^2$ and $2x^2$, gives $2x\sqrt{a^2+x^2}=3a^2-2x^2$;

Then by squaring, it is $4x^2 \times \overline{a^2 + x^2} = \overline{3a^2 - 2x^2}^2$;
That is, $4a^2x^2 + 4x^4 = 9a^4 - 12a^2x^2 + 4x^4$;
By taking $4x^4$ from both sides, it is $4a^2x^2 = 9a^4 - 12a^2x^2$;
Then transposing $12a^2x^2$, gives $16a^2x^2 = 4a^4$;
Dividing by a^2, gives $16x^2 = 4a^2$;
And dividing by 16, gives $x^2 = \frac{4}{16}a^2$;
Lastly extracting the root, gives $x = \frac{1}{2}a$.

EXAMPLES FOR PRACTICE.

1. Given $2x - 5 + 16 = 21$; to find x. Ans. $x = 5$.
2. Given $9x - 15 = x + 6$; to find x. Ans. $x = 4\frac{1}{2}$.
3. Given $8 - 3x + 12 = 30 - 5x + 4$; to find x. Ans. $x = 7$.
4. Given $x + \frac{1}{3}x - \frac{1}{4}x = 13$; to find x. Ans. $x = 12$.
5. Given $3x + \frac{1}{2}x + 2 = 5x - 4$; to find x. Ans. $x = 4$.
6. Given $4ax + \frac{1}{3}a - 2 = ax - bx$; to find x.

   Ans. $x = \dfrac{6 - a}{9a + 3b}$.
7. Given $\frac{1}{3}x - \frac{1}{4}x + \frac{1}{5}x = \frac{1}{6}$; to find x. Ans. $x = \frac{10}{17}$.
8. Given $\sqrt{4 + x} = 4 - \sqrt{x}$; to find x. Ans. $x = 2\frac{1}{4}$.
9. Given $4a + x = \dfrac{x^2}{4a + x}$; to deter. x. Ans. $x = -2a$.
10. Given $\sqrt{4a^2 + x^2} = \sqrt[4]{4b^4 + x^4}$; to find x.

    Ans. $x = \sqrt{\dfrac{b^4 - 4a^4}{2a^2}}$.
11. Given $\sqrt{x} + \sqrt{2a + x} = \dfrac{4a}{\sqrt{2a + x}}$; to find x.

    Ans. $x = \frac{1}{3}a$.
12. Given $\dfrac{a}{1 + 2x} + \dfrac{a}{1 - 2x} = 2b$; to find x.

    Ans. $x = \frac{1}{2}\sqrt{\dfrac{b - a}{b}}$.
13. Given $a + x = \sqrt{a^2 + x\sqrt{4b^2 + x^2}}$; to find x.

    Ans. $x = \dfrac{b^2}{a} - a$.

OF REDUCING DOUBLE, TRIPLE, &c. EQUATIONS, CONTAINING TWO, THREE, OR MORE UNKNOWN QUANTITIES.

PROBLEM I.

To Exterminate Two Unknown Quantities; Or, to Reduce the Two Simple Equations containing them, to a Single one.

RULE I.

FIND the value of one of the unknown letters, in terms of the other quantities, in each of the equations, by the methods already explained. Then put those two values equal to each other for a new equation, with only one unknown quantity in it, whose value is to be found as before.

Note. It is evident that we must first begin to find the values of that letter which are easiest to be found in the two proposed equations.

EXAMPLES.

1. Given $\left\{\begin{array}{l} 2x + 3y = 17 \\ 5x - 2y = 14 \end{array}\right\}$; to find x and y.

In the 1st equat. transp. $3y$ and div. by 2, gives $x = \frac{17-3y}{2}$;

In the 2d transp. $2y$ and div. by 5, gives $x = \frac{14+2y}{5}$;

Putting these two values equal, gives $\frac{14+2y}{5} = \frac{17-3y}{2}$;

Then mult. by 5 and 2, gives $28 + 4y = 85 - 15y$;
Transposing 28 and $15y$, gives $19y = 57$;
And dividing by 19, gives $y = 3$.
And hence $x = 4$.

Or, to do the same by finding two values of y, thus:

In the 1st equat. tr. $2x$ and div. by 3, gives $y = \frac{17-2x}{3}$;

In the 2d tr. $2y$ and 14, and div. by 2, gives $y = \frac{5x-14}{2}$;

Putting these two values equal, gives $\frac{5x-14}{2} = \frac{17-2x}{3}$;

Mult. by 2 and by 3, gives $15x - 42 = 34 - 4x$;

Transp. 42 and $4x$, gives $19x = 76$;
Dividing by 19, gives $x = 4$.
Hence $y = 3$, as before.

2. Given $\begin{cases} \frac{1}{2}x + 2y = a \\ \frac{1}{2}x - 2y = b \end{cases}$; to find x and y.
Ans. $x = a + b$, and $y = \frac{1}{4}a - \frac{1}{4}b$.

3. Given $3x + y = 22$, and $3y + x = 18$; to find x and y.
Ans. $x = 6$, and $y = 4$.

4. Given $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = 4 \\ \frac{1}{3}x + \frac{1}{2}y = 3\frac{1}{2} \end{cases}$; to find x and y.
Ans. $x = 6$, and $y = 3$.

5. Given $\frac{2x}{3} + \frac{3y}{5} = \frac{22}{5}$, and $\frac{3x}{5} + \frac{2y}{3} = \frac{67}{15}$; to find x and y.
Ans. $x = 3$, and $y = 4$.

6. Given $x + 2y = s$, and $x^2 - 4y^2 = d^2$; to find x and y.
Ans. $x = \frac{s^2 + d^2}{2s}$, and $y = \frac{s^2 - d^2}{4s}$.

7. Given $x - 2y = d$, and $x : y :: a : b$; to find x and y.
Ans. $x = \frac{ad}{a - 2b}$, and $y = \frac{bd}{a - 2b}$.

RULE II.

Find the value of one of the unknown letters, in only one of the equations, as in the former rule; and substitute this value instead of that unknown quantity in the other equation, and there will arise a new equation, with only one unknown quantity, whose value is to be found as before.

Note. It is evident that it is best to begin first with that letter whose value is easiest found in the given equations.

EXAMPLES.

1. Given $\begin{cases} 2x + 3y = 17 \\ 5x - 2y = 14 \end{cases}$; to find x and y.

This will admit of four ways of solution; thus: First, in the 1st eq. trans. $3y$ and div. by 2, gives $x = \frac{17 - 3y}{2}$;

This val. subs. for x in the 2d, gives $\frac{85 - 15y}{2} - 2y = 14$;
Mult. by 2, this becomes $85 - 15y - 4y = 28$;

Transp.

Transp. $15y$ and $4y$ and 28, gives $57 = 19y$;
And dividing by 19, gives $3 = y$.
Then $x = \frac{17-3y}{2} = 4$.

2dly, in the 2d trans. $2y$ and div. by 5, gives $x = \frac{14+2y}{5}$;
This subst. for x in the 1st, gives $\frac{28+4y}{5} + 3y = 17$;
Mult. by 5, gives $28 + 4y + 15y = 85$;
Transpos. 28, gives $19y = 57$;
And dividing by 19, gives $y = 3$.
Then $x = \frac{14+2y}{5} = 4$, as before.

3dly, in the 1st trans. $2x$ and div. by 3, gives $y = \frac{17-2x}{3}$;
This subst. for y in the 2d, gives $5x - \frac{34-4x}{3} = 14$;
Multiplying by 3 gives $15x - 34 + 4x = 42$;
Transposing 34, gives $19x = 76$;
And dividing by 19, gives $x = 4$.
Hence $y = \frac{17-2x}{3} = 3$, as before.

4thly, in the 2d tr. $2y$ and 14 and div. by 2, gives $y = \frac{5x-14}{2}$;
This substituted in the 1st, gives $2x + \frac{15x-42}{2} = 17$;
Multiplying by 2, gives $19x - 42 = 34$;
Transposing 42, gives $19x = 76$;
And dividing by 19, gives $x = 4$.
Hence $y = \frac{5x-14}{2} = 3$, as before.

2. Given $2x + 3y = 29$, and $3x - 2y = 11$; to find x and y. Ans. $x = 7$, and $y = 5$.

3. Given $\left\{ \begin{array}{l} x + y = 14 \\ x - y = 2 \end{array} \right\}$; to find x and y.

Ans. $x = 8$, ...

4. Given $\begin{cases} x : y :: 3 : 2 \\ x^2 - y^2 = 20 \end{cases}$; to find x and y.

Ans. $x = 6$, and $y = 4$.

5. Given $\frac{x}{3} + 3y = 21$, and $\frac{y}{3} + 3x = 29$; to find x and y. Ans. $x = 9$, and $y = 6$.

6. Given $10 - \frac{x}{2} = \frac{y}{3} + 4$, and $\frac{x-y}{2} + \frac{x}{4} - 2 = \frac{3y-x}{5} - 1$; to find x and y. Ans. $x = 8$, and $y = 6$.

7. Given $x : y :: 4 : 3$, and $x^3 - y^3 = 37$; to find x and y.

Ans. $x = 4$, and $y = 3$.

RULE III.

Let the given equations be so multiplied, or divided, &c, and by such numbers or quantities, as will make the terms which contain one of the unknown quantities the same in both equations; if they are not the same when first proposed.

Then by adding or subtracting the equations, according as the signs may require, there will remain a new equation, with only one unknown quantity, as before. That is, add the two equations when the signs are unlike, but subtract them when the signs are alike, to cancel that common term.

Note. To make two unequal terms become equal, as above, multiply each term by the co-efficient of the other.

EXAMPLES.

Given $\begin{cases} 5x - 3y = 9 \\ 2x + 5y = 16 \end{cases}$; to find x and y.

Here we may either make the two first terms, containing x, equal, or the two 2d terms, containing y, equal. To make the two first terms equal, we must multiply the 1st equation by 2, and the 2d by 5; but to make the two 2d terms equal, we must multiply the 1st equation by 5, and the 2d by 3; as follows.

1. By

1. By making the two first terms equal:
Mult. the 1st equ. by 2, gives $10x - 6y = 18$;
And mult. the 2d by 5, gives $10x + 25y = 80$;
Subtr. the upper from the under, gives $31y = 62$;
And dividing by 31, gives $y = 2$.
Hence, from the 1st given equ. $x = \frac{9 + 3y}{5} = 3$.

2. By making the two 2d terms equal:
Mult. the 1st equat. by 5, gives $25x - 15y = 45$;
And mult. the 2d by 3, gives $6x + 15y = 48$;
Adding these two, gives $31x = 93$;
And dividing by 31, gives $x = 3$.
Hence, from the 1st equ. $y = \frac{5x - 9}{3} = 2$.

MISCELLANEOUS EXAMPLES.

1. Given $\frac{x+8}{4} + 6y = 21$, and $\frac{y+6}{3} + 5x = 23$; to find x and y. Ans. $x = 4$, and $y = 3$.

2. Given $\frac{3x - y}{4} + 10 = 13$, and $\frac{3y + x}{2} + 5 = 12$; to find x and y. Ans. $x = 5$, and $y = 3$.

3. Given $\frac{3x + 4y}{5} + \frac{x}{4} = 10$, and $\frac{6x - 2y}{3} + \frac{y}{6} = 14$; to find x and y. Ans. $x = 8$, and $y = 4$.

4. Given $3x + 4y = 38$, and $4x - 3y = 9$; to find x and y.
Ans. $x = 6$, and $y = 5$.

PROBLEM II.

To Exterminate Three or More Unknown Quantities; Or, to Reduce the Simple Equations, containing them, to a Single one.

RULE.

This may be done by any of the three methods in the last problem: viz.

1. After the manner of the first rule in the last problem, find the value of one of the unknown letters in each of the given equations: next put two of these values equal to each other, and then one of these and a third value equal, and so on for all the values of it; which gives a new set of equations, with

with which the same process is to be repeated, and so on till there is only one equation, to be reduced by the rules for a single equation.

2. Or, as in the 2d rule of the same problem, find the value of one of the unknown quantities in one of the equations only; then substitute this value instead of it in the other equations; which gives a new set of equations to be resolved as before, by repeating the operation.

3. Or, as in the 3d rule, reduce the equations, by multiplying or dividing them, so as to make some of the terms to agree: then, by adding or subtracting them, as the signs may require, one of the letters may be exterminated, &c, as before.

EXAMPLES.

1. Given $\begin{cases} x + y + z = 9 \\ x + 2y + 3z = 16 \\ x + 3y + 4z = 21 \end{cases}$; to find x, y, and z.

1. By the 1st method:

Transp. the terms containing y and z in each equa. gives

$$x = 9 - y - z,$$
$$x = 16 - 2y - 3z,$$
$$x = 21 - 3y - 4z;$$

Then putting the 1st and 2d values equal, and the 2d and 3d values equal, give

$$9 - y - z = 16 - 2y - 3z,$$
$$16 - 2y - 3z = 21 - 3y - 4z;$$

In the 1st trans. 9, z, and $2y$, gives $y = 7 - 2z$;
In the 2d trans. 16, $3z$, and $3y$, gives $y = 5 - z$;
Putting these two equal, gives $5 - z = 7 - 2z$;
Trans. 5 and $2z$, gives $z = 2$.
Hence $y = 5 - z = 3$, and $x = 9 - y - z = 4$.

2dly. By the 2d method:

From the 1st equa. $x = 9 - y - z$;
This value of x substit. in the 2d and 3d, gives

$$9 + y + 2z = 16,$$
$$9 + 2y + 3z = 21;$$

In the 1st trans. 9 and $2z$, gives $y = 7 - 2z$;
This substit. in the last, gives $23 - z = 21$;
Trans. z and 21, gives $2 = z$.
Hence again $y = 7 - 2z = 3$, and $x = 9 - y - z = 4$.

3dly. By

3dly. By the 3d method: subtracting the 1st equ. from the 2d, and the 2d from the 3d, gives

$$y + 2z = 7,$$
$$y + z = 5;$$

Subtr. the latter from the former, gives $z = 2$.

Hence $y = 5 - z = 3$, and $x = 9 - y - z = 4$.

2. Given $\left\{\begin{array}{l} x + y + z = 18 \\ x + 3y + 2z = 38 \\ x + \frac{1}{3}y + \frac{1}{2}z = 10 \end{array}\right\}$; to find x, y, and z.

Ans. $x = 4$, $y = 6$, $z = 8$.

3. Given $\left\{\begin{array}{l} x + \frac{1}{2}y + \frac{1}{3}z = 27 \\ x + \frac{1}{3}y + \frac{1}{4}z = 20 \\ x + \frac{1}{4}y + \frac{1}{5}z = 16 \end{array}\right\}$; to find x, y, and z.

Ans. $x = 1$, $y = 20$, $z = 60$.

4. Given $x - y = 2$, $x - z = 3$, and $y - z = 1$; to find x, y, and z. Ans. $x = 7$; $y = 5$; $z = 4$.

5. Given $\left\{\begin{array}{l} 2x + 3y + 4z = 34 \\ 3x + 4y + 5z = 46 \\ 4x + 5y + 6z = 58 \end{array}\right\}$; to find x, y, and z.

A COLLECTION OF QUESTIONS PRODUCING SIMPLE EQUATIONS.

QUEST. 1. To find two numbers, such, that their sum shall be 10, and their difference 6.

Let x denote the greater number, and y the less *.

Then, by the 1st condition $x + y = 10$,

And by the 2d - - $x - y = 6$,

Transp. y in each, gives $x = 10 - y$,

and $x = 6 + y$;

Put these two values equal, gives $6 + y = 10 - y$;

Transpos. 6 and $-y$, gives - $2y = 4$;

Dividing by 2, gives - - $y = 2$.

And hence - - - - $x = 6 + y = 8$.

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* In all these solutions, as many unknown letters are always used as there are unknown numbers to be found, purposely the better to exercise the modes of reducing the equations: avoiding the short ways of notation, which, though giving a shorter solution, are for that reason less useful to the pupil, as affording less exercise in practising the several rules in reducing equations.

QUEST. 2

QUEST. 2. Divide 100*l.* among A, B, C, so that A may have 20*l.* more than B, and B 10*l.* more than C.

Let x = A's share, y = B's, and z = C's.

$$\text{Then } x + y + z = 100,$$
$$x = y + 20,$$
$$y = z + 10.$$

In the 1st substit. $y + 20$ for x, gives $2y + z + 20 = 100$;
In this substituting $z + 10$ for y, gives $3z + 40 = 100$;
By transposing 40, gives - $3z = 60$;
And dividing by 3, gives - - $z = 20$.
Hence $y = z + 10 = 30$, and $x = y + 20 = 50$.

QUEST. 3. A prize of 500*l.* is to be divided between two persons, so as their shares may be in proportion as 7 to 8; required the share of each.

Put x and y for the two shares; then by the question,
$7 : 8 :: x : y$, or mult. the extremes and the means, $7y = 8x$,
and $x + y = 500$;
Transposing y, gives $x = 500 - y$;
This substituted in the 1st, gives $7y = 4000 - 8y$;
By transposing $8y$, it is $15y = 4000$;
By dividing by 15, it gives $y = 266\frac{2}{3}$;
And hence $x = 500 - y = 233\frac{1}{3}$.

QUEST. 4. What number is that whose 4th part exceeds its 5th part by 10?

Let x denote the number sought.
Then by the question $\frac{1}{4}x - \frac{1}{5}x = 10$;
By mult. by 4, it becomes $x - \frac{4}{5}x = 40$;
By mult. by 5, it gives $x = 200$, the number sought.

QUEST. 5. What fraction is that, to the numerator of which if 1 be added, the value will be $\frac{1}{2}$; but if 1 be added to the denominator, its value will be $\frac{1}{3}$?

Let $\frac{x}{y}$ denote the fraction.

Then by the quest. $\frac{x+1}{y} = \frac{1}{2}$, and $\frac{x}{y+1} = \frac{1}{3}$.
The 1st mult. by 2 and y, gives $2x + 2 = y$;
The 2d mult. by 3 and $y + 1$, is $3x = y + 1$;
The upper taken from the under leaves $x - 2 = 1$;
By transpos. 2, it gives $x = 3$.
And hence $y = 2x + 2 = 8$; and the fraction is $\frac{3}{8}$.

QUEST. 6.

QUEST. 6. A labourer engaged to serve for 30 days on these conditions: that for every day he worked, he was to receive 20*d.* but for every day he played, or was absent, he was to forfeit 10*d.* Now at the end of the time he had to receive just 20 shillings, or 240 pence. It is required to find how many days he worked, and how many he was idle?

Let x be the days worked, and y the days idled.
Then $20x$ is the pence earned, and $10y$ the forfeits;
Hence, by the question - $x + y = 30$,
and $20x - 10y = 240$;
The 1st. mult. by 10, gives $10x + 10y = 300$;
These two added give - $30x = 540$;
This div. by 30, gives - $x = 18$, the days worked;
Hence - $y = 30 - x = 12$, the days idled.

QUEST. 7. Out of a cask of wine, which had leaked away $\frac{1}{4}$, 30 gallons were drawn; and then, being gaged, it appeared to be half full; how much did it hold?

Let it be supposed to have held x gallons,
Then it would have leaked $\frac{1}{4}x$ gallons,
Conseq. there had been taken away $\frac{1}{4}x + 30$ gallons.
Hence $\frac{1}{2}x = \frac{1}{4}x + 30$ by the question.
Then mult. by 4, gives $2x = x + 120$;
And transposing x, gives $x = 120$ the contents.

QUEST. 8. To divide 20 into two such parts, that 3 times the one part added to 5 times the other may make 76.

Let x and y denote the two parts.
Then by the question - - $x + y = 20$,
and $3x + 5y = 76$.
Mult. the 1st by 3, gives - $3x + 3y = 60$;
Subtr. the latter from the former, gives $2y = 16$;
And dividing by 2, gives - - $y = 8$.
Hence, from the 1st, - $x = 20 - y = 12$.

QUEST. 9. A market woman bought in a certain number of eggs at 2 a penny, and as many more at 3 a penny, and sold them all out again at the rate of 5 for two-pence, and by so doing, contrary to expectation, found she lost 3*d.*; what number of eggs had she?

Let x = number of eggs of each sort.
Then will $\frac{1}{2}x$ = cost of the first sort,
And $\frac{1}{3}x$ = cost of the second sort;

But

But $5 : 2 :: 2x$ (the whole number of eggs) : $\frac{4}{5}x$;
Hence $\frac{4}{5}x =$ price of both sorts, at 5 for 2 pence;
Then by the question $\frac{1}{2}x + \frac{1}{3}x - \frac{4}{5}x = 3$;
Mult. by 2, gives - $x + \frac{2}{3}x - \frac{8}{5}x = 6$;
And mult. by 3, gives $5x - \frac{24}{5}x = 18$;
Also mult. by 5, gives $x = 90$, the number of eggs of each sort.

QUEST. 10. Two persons, A and B, engage at play. Before they begin, A has 80 guineas, and B has 60. After a certain number of games won and lost between them, A rises with three times as many guineas as B. Query, how many guineas did A win of B?

Let x denote the number of guineas A won.
Then A rises with $80 + x$,
And B rises with $60 - x$;
Theref. by the quest. $80 + x = 180 - 3x$;
Transp. 80 and $3x$, gives $4x = 100$;
And dividing by 4, gives $x = 25$, the guineas won.

QUESTIONS FOR PRACTICE.

1. To determine two numbers such, that their difference may be 4, and the difference of their squares 64.
Ans. 6 and 10.

2. To find two numbers with these conditions, viz. that half the first with a 3d part of the second may make 9, and that a 4th part of the first with a 5th part of the second may make 5. Ans. 8 and 15.

3. To divide the number 20 into two such parts, that a 3d of the one part added to a fifth of the other, may make 6. Ans. 15 and 5.

4. To find three numbers such, that the sum of the 1st and 2d shall be 7, the sum of the 1st and 3d 8, and the sum of the 2d and 3d 9. Ans. 3, 4, 5.

5. A father, dying, bequeathed his fortune, which was 2800*l.* to his son and daughter, in this manner; that for every half crown the son might have, the daughter was to have a shilling. What then were their two shares?
Ans. The son 2000*l.* and the daughter 800*l.*

6. Three persons, A, B, C, make a joint contribution, which in the whole amounts to 400*l.*: of which sum B contrib

tributes twice as much as A and 20*l.* more; and C as much as A and B together. What sum did each contribute?

Ans. A 60*l.* B 140*l.* and C 200*l.*

7. A person paid a bill of 100*l.* with half guineas and crowns, using in all 202 pieces; how many pieces were there of each sort?

Ans. 180 half guineas, and 22 crowns.

8. Says A to B, if you give me 10 guineas of your money, I shall then have twice as much as you will have left: but says B to A, give me 10 of your guineas, and then I shall have 3 times as many as you. How many had each?

Ans. A 22, B 26.

9. A person goes to a tavern with a certain quantity of money in his pocket, where he spends 2 shillings; he then borrows as much money as he had left, and going to another tavern, he there spends 2 shillings also; then borrowing again as much money as was left, he went to a third tavern, where likewise he spent 2 shillings; and thus repeating the same at a fourth tavern, he then had nothing remaining. What sum had he at first? Ans. 3*s.* 9*d.*

10. A man with his wife and child dine together at an inn. The landlord charged 1 shilling for the child; and for the woman he charged as much as for the child and $\frac{1}{4}$ as much as for the man; and for the man he charged as much as for the woman and child together. How much was that for each? Ans. The woman 20*d.* and the man 32*d.*

11. A cask, which held 60 gallons, was filled with a mixture of brandy, wine, and cyder, in this manner, viz. the cyder was 6 gallons more than the brandy, and the wine was as much as the cyder and $\frac{1}{5}$ of the brandy. How much was there of each?

Ans. Brandy 15, cyder 21, wine 24.

12. A general, disposing his army into a square form, finds that he has 284 men more than a perfect square; but increasing the side by 1 man, he then wants 25 men to be a complete square. Then how many men had he under his command? Ans. 24000.

13. What number is that, to which if 3, 5, and 8, be severally added, the three sums shall be in geometrical progression? Ans. 1.

13. The [illegible] ders amounted to 860*l.* the shares of [illegible] ded that of the third by

by 240; and the sum of the 2d and 3d exceeded the first by 260. What was the share of each?

Ans. The 1st 200, the 2d 300, the 3d 260.

15. What two numbers are those, which, being in the ratio of 3 to 4, their product is equal to 12 times their sum?

Ans. 21 and 28.

16. A certain company at a tavern, when they came to settle their reckoning, found that had there been 4 more in company, they might have paid a shilling a-piece less than they did; but that if there had been 3 fewer in company, they must have paid a shilling a-piece more than they did. What then was the number of persons in company, what each paid, and what was the whole reckoning?

Ans. 24 persons, each paid 7*s.* and the whole reckoning 8 guineas.

17. A jockey has two horses; and also two saddles, the one valued at 18*l.* the other at 3*l.* Now when he sets the better saddle on the 1st horse, and the worse on the 2d, it makes the first horse worth double the 2d: but when he places the better saddle on the 2d horse, and the worse on the first, it makes the 2d horse worth three times the 1st. What then were the values of the two horses?

Ans. The 1st 6*l.*, and the 2d 9*l.*

18. What two numbers are as 2 to 3, to each of which if 6 be added, the sums will be as 4 to 5?

Ans. 6 and 9.

19. What are those two numbers, of which the greater is to the less as their sum is to 20, and as their difference is to 10? Ans. 15 and 45.

20. What two numbers are those, whose difference, sum, and product, are to each other, as the three numbers 2, 3, 5? Ans. 2 and 10.

21. To find three numbers in arithmetical progression, of which the first is to the third as 5 to 9, and the sum of all three is 63. Ans. 15, 21, 27.

22. It is required to divide the number 24 into two such parts, that the quotient of the greater part divided by the less, may be to the quotient of the less part divided by the greater, as 4 to 1. Ans. 16 and 8.

23. A gentleman being asked the age of his two sons, answered, that if to the sum of their ages 18 be add the result will be double the age of the elder; but

taken from the difference of their ages, the remainder will be equal to the age of the younger. What then were their ages? Ans. 30 and 12.

24. To find four numbers such, that the sum of the 1st, 2d, and 3d shall be 13; the sum of the 1st, 2d, and 4th, 15; the sum of the 1st, 3d, and 4th, 18; and lastly the sum of the 2d, 3d, and 4th, 20. Ans. 2, 4, 7, 9.

25. To divide 48 into 4 such parts, that the first increased by 3, the second diminished by 3, the third multiplied by 3, and the 4th divided by 3, may be all equal to each other.
Ans. 6, 12, 3, 27.

QUADRATIC EQUATIONS.

Quadratic Equations are either simple or compound.

A simple quadratic equation, is that which involves the square of the unknown quantity only. As $ax^2 = b$. And the solution of such quadratics has been already given in simple equations.

A compound quadratic equation, is that which contains the square of the unknown quantity in one term, and the first power in another term. As $ax^2 + bx = c$.

All compound quadratic equations, after being properly reduced, fall under the three following forms, to which they must always be reduced by preparing them for solution.

$$1. \quad x^2 + ax = b$$
$$2. \quad x^2 - ax = b$$
$$3. \quad x^2 - ax = -b$$

The general method of solving quadratic equations, is by what is called completing the square, which is as follows:

1. Reduce the proposed equation to a proper simple form, as usual, such as the forms above; namely, by transposing all the terms which contain the unknown quantity to one side of the equation, and the known terms to the other; placing the square term first, and the single power second; dividing the equation by the co-efficient of the square or first term, if it has one, and changing the signs of all the terms, when that term happens to be negative, as that term must always be made positive before the solution. Then the is by completing the square as

2. Complete

2. Complete the unknown side to a square, in this manner, viz. Take half the co-efficient of the second term, and square it; which square add to both sides of the equation, then that side which contains the unknown quantity will be a complete square.

3. Then extract the square root on both sides of the equation*, and the value of the unknown quantity will be determined,

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* As the square root of any quantity may be either + or −, therefore all quadratic equations admit of two solutions. Thus, the square root of $+n^2$ is either $+n$ or $-n$; for $+n \times +n$ and $-n \times -n$ are each equal to $+n^2$. But the square root of $-n^2$, or $\sqrt{-n^2}$, is imaginary or impossible, as neither $+n$ nor $-n$, when squared, gives $-n^2$.

So, in the first form, $x^2 + ax = b$, where $x + \frac{1}{2}a$ is found $= \sqrt{b + \frac{1}{4}a^2}$, the root may be either $+\sqrt{b + \frac{1}{4}a^2}$, or $-\sqrt{b + \frac{1}{4}a^2}$, since either of them being multiplied by itself produces $b + \frac{1}{4}a^2$. And this ambiguity is expressed by writing the uncertain or double sign $\pm$ before $\sqrt{b + \frac{1}{4}a^2}$; thus $x = \pm\sqrt{b + \frac{1}{4}a^2} - \frac{1}{2}a$.

In this form, where $x = \pm\sqrt{b + \frac{1}{4}a^2} - \frac{1}{2}a$, the first value of x, viz. $x = +\sqrt{b + \frac{1}{4}a^2} - \frac{1}{2}a$, is always affirmative; for since $\frac{1}{4}a^2 + b$ is greater than $\frac{1}{4}a^2$, the greater square must necessarily have the greater root; therefore $\sqrt{b + \frac{1}{4}a^2}$ will always be greater than $\sqrt{\frac{1}{4}a^2}$, or its equal $\frac{1}{2}a$; and consequently $+\sqrt{b + \frac{1}{4}a^2} - \frac{1}{2}a$ will always be affirmative.

The second value, viz. $x = -\sqrt{b + \frac{1}{4}a^2} - \frac{1}{2}a$ will always be negative, because it is composed of two negative terms. Therefore when $x^2 + ax = b$, we shall have $x = +\sqrt{b + \frac{1}{4}a^2} - \frac{1}{2}a$ for the affirmative value of x, and $x = -\sqrt{b + \frac{1}{2}a^2} - \frac{1}{2}a$ for the negative value of x.

In the second form, where $x = \pm\sqrt{b + \frac{1}{4}a^2} + \frac{1}{2}a$ the first value, viz. $x = +\sqrt{b + \frac{1}{4}a^2} + \frac{1}{2}a$ is always affirmative, since it is composed of two affirmative terms. But the second value, viz. $x = -\sqrt{b + \frac{1}{4}a^2} + \frac{1}{2}a$, will always be negative; for since $b + \frac{1}{4}a^2$ is greater than $\frac{1}{4}a^2$, therefore $\sqrt{b + \frac{1}{4}a^2}$ will be greater than $\sqrt{\frac{1}{4}a^2}$, or its equal $\frac{1}{2}a$; and consequently $-\sqrt{b + \frac{1}{4}a^2} + \frac{1}{2}a$ is always a negative quantity.

Therefore,

determined, making the root of the known side either + or −, which will give two roots of the equation, or two values of the unknown quantity.

Note, 1. The root of the first side of the equation, is always equal to the root of the first term, with half the co-efficient of the second term joined to it, with its sign, whether + or −.

2. All equations, in which there are two terms including the unknown quantity, and which have the index of the one just double that of the other, are resolved like quadratics, by completing the square, as above.

Thus, $x^4 + ax^2 = b$, or $x^{2n} + ax^n = b$, or $x + ax^{\frac{1}{2}} = b$, are the same as quadratics, and the value of the unknown quantity may be determined accordingly.

Therefore, when $x^2 - ax = b$, we shall have $x = + \sqrt{b + \frac{1}{4}a^2} + \frac{1}{2}a$ for the affirmative value of x; and $x = - \sqrt{b + \frac{1}{4}a^2} + \frac{1}{2}a$ for the negative value of x; so that in both the first and second forms, the unknown quantity has always two values, one of which is positive, and the other negative.

But, in the third form, where $x = \pm \sqrt{\frac{1}{4}a^2 - b} + \frac{1}{2}a$, both the values of x will be positive, when $\frac{1}{4}a^2$ is greater than b. For the first value, viz. $x = + \sqrt{\frac{1}{4}a^2 - b} + \frac{1}{2}a$ will then be affirmative, being composed of two affirmative terms.

The second value, viz. $x = - \sqrt{\frac{1}{4}a^2 - b} + \frac{1}{2}a$ is affirmative also; for since $\frac{1}{4}a^2$ is greater than $\frac{1}{4}a^2 - b$, therefore $\sqrt{\frac{1}{4}a^2}$ or $\frac{1}{2}a$ is greater than $\sqrt{\frac{1}{4}a^2 - b}$; and consequently $- \sqrt{\frac{1}{4}a^2 - b} + \frac{1}{2}a$ will always be an affirmative quantity. So that, when $x^2 - ax = - b$, we shall have $x = + \sqrt{\frac{1}{4}a^2 - b} + \frac{1}{2}a$, and also $x = - \sqrt{\frac{1}{4}a^2 - b} + \frac{1}{2}a$, for the values of x, both positive.

But in this third form, if b be greater than $\frac{1}{4}a^2$, the solution of the proposed question will be impossible. For since the square of any quantity (whether that quantity be affirmative or negative) is always affirmative, the square root of a negative quantity is impossible, and cannot be assigned. But when b is greater than $\frac{1}{4}a^2$, then $\frac{1}{4}a^2 - b$ is a negative quantity; and therefore its root $\sqrt{\frac{1}{4}a^2 - b}$ is impossible, or imaginary; consequently, in that case, $x = \frac{1}{2}a \pm \sqrt{\frac{1}{4}a^2 - b}$, or the two roots or values of x, are both impossible, or imaginary quantities.

EXAMPLES.

1. Given $x^2 + 4x = 60$; to find x.

First, by completing the square, $x^2 + 4x + 4 = 64$;
Then, by extracting the root, $x + 2 = \pm 8$;
Then, transpos. 2, gives $x = 6$ or -10, the two roots.

2. Given $x^2 - 6x + 10 = 65$; to find x.

First, trans. 10, gives $x^2 - 6x = 55$;
Then by complet. the sq. it is $x^2 - 6x + 9 = 64$;
And by extr. the root, gives $x - 3 = \pm 8$;
Then trans. 3, gives $x = 11$ or -5.

3. Given $2x^2 + 8x - 30 = 60$; to find x.

First by transpos. 20, it is $2x^2 + 8x = 90$;
Then div. by 2, gives $x^2 + 4x = 45$;
And by compl. the sq. it is $x^2 + 4x + 4 = 49$;
Then extr. the root, it is $x + 2 = \pm 7$;
And transp. 2, gives $x = 5$ or -9.

4. Given $3x^2 - 3x + 9 = 8\frac{1}{3}$; to find x.

First div. by 3, gives $x^2 - x + 3 = 2\frac{7}{9}$;
Then transpos. 3, gives $x^2 - x = -\frac{2}{9}$;
And compl. the sq. gives $x^2 - x + \frac{1}{4} = \frac{1}{36}$;
Then extr. the root gives $x - \frac{1}{2} = \pm \frac{1}{6}$;
And transp. $\frac{1}{2}$, gives $x = \frac{2}{3}$ or $\frac{1}{3}$.

5. Given $\frac{1}{2}x^2 - \frac{1}{3}x + 30\frac{1}{2} = 52\frac{2}{3}$; to find x.

First by transpos. $30\frac{1}{2}$, it is $\frac{1}{2}x^2 - \frac{1}{3}x = 22\frac{1}{6}$;
Then mult. by 2 gives $x^2 - \frac{2}{3}x = 44\frac{1}{3}$;
And by compl. the sq. it is $x^2 - \frac{2}{3}x + \frac{1}{9} = 44\frac{4}{9}$;
Then extr. the root, gives $x - \frac{1}{3} = \pm 6\frac{2}{3}$;
And transp. $\frac{1}{3}$, gives $x = 7$ or $-6\frac{1}{3}$.

6. Given $ax^2 - bx = c$; to find x.

First by div. by a, it is $x^2 - \dfrac{b}{a}x = \dfrac{c}{a}$;

Then compl. the sq. gives $x^2 - \dfrac{b}{a}x + \dfrac{b^2}{4a^2} = \dfrac{c}{a} + \dfrac{b^2}{4a^2}$;

And extrac. the root, gives $x - \dfrac{b}{2a} = \pm\sqrt{\dfrac{4ac + b^2}{4a^2}}$;

Then transp. $\dfrac{b}{2a}$, gives $x = \pm\sqrt{\dfrac{4ac + b^2}{4a^2}} + \dfrac{b}{2a}$.

7. Given $x^4 - 2ax^2 = b$; to find x.

First by compl. the sq. gives $x^4 - 2ax^2 + a^2 = a^2 + b$;

And

And extract. the root, gives $x^2 - a = \pm\sqrt{a^2 + b}$;
Then transpos. a, gives $x^2 = \pm\sqrt{a^2 + b} + a$;
And extract. the root, gives $x = \pm\sqrt{a \pm \sqrt{a^2 + b}}$.

And thus, by always using similar words at each line, the pupil will resolve the following examples.

EXAMPLES FOR PRACTICE.

1. Given $x^2 - 6x - 7 = 33$; to find x. Ans. $x = 10$.
2. Given $x^2 - 5x - 10 = 14$; to find x. Ans. $x = 8$.
3. Given $5x^2 + 4x - 90 = 114$; to find x. Ans. $x = 6$.
4. Given $\frac{1}{2}x^2 - \frac{1}{4}x + 2 = 9$; to find x. Ans. $x = 4$.
5. Given $3x^4 - 2x^2 = 40$; to find x. Ans. $x = 2$.
6. Given $\frac{1}{3}x - \frac{1}{2}\sqrt{x} = 1\frac{1}{2}$; to find x. Ans. $x = 9$.
7. Given $\frac{1}{2}x^2 + \frac{2}{3}x = \frac{3}{4}$; to find x. Ans. $x = \cdot 727766$.
8. Given $x^6 + 4x^3 = 12$; to find x.
   Ans. $x = \sqrt[3]{2} = 1\cdot 259921$.
9. Given $x^2 + 4x = a^2 + 2$; to find x.
   Ans. $x = \sqrt{a^2 + 6} - 2$.

QUESTIONS PRODUCING QUADRATIC EQUATIONS.

1. To find two numbers whose difference is 2, and product 80.

Let x and y denote the two required numbers*.
Then the first condition gives $x - y = 2$,
And the second gives $xy = 80$.
Then transp. y in the 1st gives $x = y + 2$;
This value of x substitut. in the 2d, is $y^2 + 2y = 80$;
Then comp. the square gives $y^2 + 2y + 1 = 81$;
And extrac. the root gives $y + 1 = 9$;
And transpos. 1 gives $y = 8$;
And therefore $x = y + 2 = 10$.

* These questions, like those in simple equations, are also solved by using as many unknown letters, as are the numbers required, for the better exercise in reducing equations; not aiming at the shortest modes of solution, which would not afford so much useful practice.

2. To divide the number 14 into two such parts, that their product may be 48.

Let x and y denote the two numbers.
Then the 1st condition gives $x + y = 14$,
And the 2d gives $xy = 48$.
Then transp. y in the 1st gives $x = 14 - y$;
This value subst. for x in the 2d, is $14y - y^2 = 48$;
Changing all the signs, to make the square positive, gives $y^2 - 14y = -48$;
Then compl. the square gives $y^2 - 14y + 49 = 1$;
And extrac. the root gives $y - 7 = \pm 1$;
Then transpos. 7, gives $y = 8$ or 6, the two parts.

3. Given the sum of two numbers $= 9$, and the sum of their squares $= 45$; to find those numbers.

Let x and y denote the two numbers.
Then by the 1st condition $x + y = 9$.
And by the 2d $x^2 + y^2 = 45$.
Then transpos. y in the 1st gives $x = 9 - y$;
This value subst. in the 2d gives $81 - 18y + 2y^2 = 45$;
Then transpos. 81, gives $2y^2 - 18y = -36$;
And dividing by 2 gives $y^2 - 9y = -18$;
Then compl. the sq. gives $y^2 - 9y + \frac{81}{4} = \frac{9}{4}$;
And extrac. the root gives $y - \frac{9}{2} = \pm \frac{3}{2}$;
Then transpos. $\frac{9}{2}$ gives $y = 6$ or 3, the two numbers.

4. What two numbers are those, whose sum, product, and difference of their squares, are all equal to each other?

Let x and y denote the two numbers.
Then the 1st and 2d expression give $x + y = xy$,
And the 1st and 3d give $x + y = x^2 - y^2$.
Then the last equa. div. by $x + y$, gives $1 = x - y$;
And transpos. y, gives $y + 1 = x$;
This val. substit. in the 1st gives $2y + 1 = y^2 + y$;
And transpos. $2y$, gives $1 = y^2 - y$;
Then complet. the sq. gives $\frac{5}{4} = y^2 - y + \frac{1}{4}$;
And extracting the root gives $\frac{1}{2}\sqrt{5} = y - \frac{1}{2}$;
And transposing $\frac{1}{2}$ gives $\frac{1}{2}\sqrt{5} + \frac{1}{2} = y$;
And therefore $x = y + 1 = \frac{1}{2}\sqrt{5} + \frac{3}{2}$.

And if these expressions be turned into numbers, by extracting the root of 5, &c, they give $x = 2{\cdot}6180 +$, and $y = 1{\cdot}6180 +$.

5. There are four numbers in arithmetical progression, of which

which the product of the two extremes is 22, and that of the means 40; what are the numbers?

Let $x =$ the less extreme,
and $y =$ the common difference;
Then x, $x+y$, $x+2y$, $x+3y$, will be the four numbers.
Hence by the 1st condition $x^2 + 3xy = 22$,
And by the 2d $x^2 + 3xy + 2y^2 = 40$.
Then subtracting the first from the 2d gives $2y^2 = 18$;
And dividing by 2 gives $y^2 = 9$;
And extracting the root gives $y = 3$.
Then substit. 3 for y in the 1st, gives $x^2 + 9x = 22$;
And completing the square gives $x^2 + 9x + \frac{81}{4} = \frac{169}{4}$;
Then extracting the root gives $x + \frac{9}{2} = \frac{13}{2}$;
And transposing $\frac{9}{2}$ gives $x = 2$ the least number.
Hence the four numbers are 2, 5, 8, 11.

6. To find 3 numbers in geometrical progression, whose sum shall be 7, and the sum of their squares 21.

Let x, y, and z denote the three numbers sought.
Then by the 1st condition $xz = y^2$,
And by the 2d $x + y + z = 7$,
And by the 3d $x^2 + y^2 + z^2 = 21$.
Transposing y in the 2d gives $x + z = 7 - y$;
Sq. this equa. gives $x^2 + 2xz + z^2 = 49 - 14y + y^2$;
Substi. $2y^2$ for $2xz$, gives $x^2 + 2y^2 + z^2 = 49 - 14y + y^2$;
Subtr. y^2 from each side, leaves $x^2 + y^2 + z^2 = 49 - 14y$;
Putting the two values of $x^2 + y^2 + z^2$ equal to each other, gives $21 = 49 - 14y$;
Then transposing 21 and $14y$, gives $14y = 28$;
And dividing by 14, gives $y = 2$.
Then substit. 2 for y in the 1st equa. gives $xz = 4$,
And in the 4th, it gives $x + z = 5$;
Transposing z in the last, gives $x = 5 - z$;
This substit. in the next above, gives $5z - z^2 = 4$;
Changing all the signs, gives $z^2 - 5z = -4$;
Then completing the square, gives $z^2 - 5z + \frac{25}{4} = \frac{9}{4}$;
And extracting the root gives $z - \frac{5}{2} = \pm \frac{3}{2}$;
Then transposing $\frac{5}{2}$, gives z and $x = 4$ and 1, the two other numbers;
So that the three numbers are 1, 2, 4.

QUESTIONS FOR PRACTICE.

1. What number is that which added to its square makes 42? Ans. 6.

2. To

2. To find two numbers such, that the less may be to the greater as the greater is to 12, and that the sum of their squares may be 45. Ans. 3 and 6.

3. What two numbers are those, whose difference is 2, and the difference of their cubes 98? Ans. 3 and 5.

4. What two numbers are those whose sum is 6, and the sum of their cubes 72? Ans. 2 and 4.

5. What two numbers are those, whose product is 20, and the difference of their cubes 61? Ans. 4 and 5.

6. To divide the number 11 into two such parts, that the product of their squares may be 784. Ans. 4 and 7.

7. To divide the number 5 into two such parts, that the sum of their alternate quotients may be $4\frac{1}{4}$, that is of the two quotients of each part divided by the other.

Ans. 1 and 4.

8. To divide 12 into two such parts, that their product may be equal to 8 times their difference. Ans. 4 and 8.

9. To divide the number 10 into two such parts, that the square of 4 times the less part, may be 112 more than the square of 2 times the greater. Ans. 4 and 6.

10. To find two numbers such, that the sum of their squares may be 89, and their sum multiplied by the greater may produce 104. Ans. 5 and 8.

11. What number is that, which being divided by the product of its two digits, the quotient is $5\frac{1}{3}$; but when 9 is subtracted from it, there remains a number having the same digits inverted? Ans. 32.

12. To divide 20 into three parts such, that the continual product of all three may be 270, and that the difference of the first and second may be 2 less than the difference of the second and third. Ans. 5, 6, 9.

13. To find three numbers in arithmetical progression, such that the sum of their squares may be 56, and the sum arising by adding together 3 times the first and 2 times the second and 3 times the third, may amount to 28.

Ans. 2, 4, 6.

14. To divide the number 13 into three such parts, that their squares may have equal differences, and that the sum of those squares may be 75. Ans. 1, 5, 7.

15. To find three numbers having equal differences, so that their sum may be 12, and the sum of their fourth powers ? Ans. 3, 4, 5.

16. To

16. To find three numbers having equal differences, and such that the square of the least added to the product of the two greater may make 28, but the square of the greatest added to the product of the two less may make 44.

Ans. 2, 4, 6.

17. Three merchants, A, B, C, on comparing their gains find, that among them all they have gained 1444*l.*; and that B's gain added to the square root of A's made 920*l.*; but if added to the square root of C's it made 912. What were their several gains? Ans. A 400, B 900, C 144.

18. To find three numbers in arithmetical progression, so that the sum of their squares shall be 93; also if the first be multiplied by 3, the second by 4, and the third by 5, the sum of the products may be 66. Ans. 2, 5, 8.

19. To find four numbers such, that the first may be to the second as the third to the fourth; and that the first may be to the fourth as 1 to 5; also the second to the third as 5 to 9; and the sum of the second and fourth may be 20.

Ans. 3, 5, 9, 15.

20. To find two numbers such, that their product added to their sum may make 47, and their sum taken from the sum of their squares may leave 62. Ans. 5 and 7.

RESOLUTION OF CUBIC AND HIGHER EQUATIONS.

A Cubic Equation, or Equation of the 3d degree or power, is one that contains the third power of the unknown quantity. As $x^3 - ax^2 + bx = c$.

A *Biquadratic*, or Double Quadratic, is an equation that contains the 4th power of the unknown quantity:

As $x^4 - ax^3 + bx^2 - cx = d$.

An Equation of the 5th Power or Degree, is one that contains the 5th power of the unknown quantity:

As $x^5 - ax^4 + bx^3 - cx^2 + dx = e$.

And so on, for all other higher powers. Where it is to be noted, however, that all the powers, or terms, in the equation, are supposed to be freed from surds or fractional exponents.

There are many particular and prolix rules usually given for the solution of some of the above-mentioned powers

or equations. But they may be all readily solved by the following easy rule of Double Position, sometimes called Trial-and-Error.

RULE.

1. FIND, by trial, two numbers, as near the true root as you can, and substitute them separately in the given equation, instead of the unknown quantity; and find how much the terms collected together, according to their signs + or −, differ from the absolute known term of the equation, marking whether these errors are in excess or defect.

2. Multiply the difference of the two numbers, found or taken by trial, by either of the errors, and divide the product by the difference of the errors, when they are alike, but by their sum when they are unlike. Or say, As the difference or sum of the errors, is to the difference of the two numbers, so is either error to the correction of its supposed number.

3. Add the quotient, last found, to the number belonging to that error, when its supposed number is too little, but subtract it when too great, and the result will give the true root *nearly*.

4. Take this root and the nearest of the two former, or any other that may be found nearer; and, by proceeding in like manner as above, a root will be had still nearer than before. And so on to any degree of exactness required.

Note 1. It is best to employ always two assumed numbers that shall differ from each other only by unity in the last figure on the right hand; because then the difference, or multiplier, is only 1. It is also best to use always the least error in the above operation.

Note 2. It will be convenient also to begin with a single figure at first, trying several single figures till there be found the two nearest the truth, the one too little, and the other too great; and in working with them, find only one more figure. Then substitute this corrected result in the equation, for the unknown letter, and if the result prove too little, substitute also the number next greater for the second supposition; but contrarywise, if the former prove too great, then take the next less number for the second supposition; and in working with the second pair of errors, continue the quotient only so far as to have the corrected number to four places of figures. Then repeat the same process again with this last corrected number, and the next greater or less, as

the

the case may require, carrying the third corrected number to eight figures; because each new operation commonly doubles the number of true figures. And thus proceed to any extent that may be wanted.

EXAMPLES.

Ex. 1. To find the root of the cubic equation $x^3 + x^2 + x = 100$, or the value of x in it.

Here it is soon found that x lies between 4 and 5. Assume therefore these two numbers, and the operation will be as follows:

1st Sup.				2d Sup.
4	-	x	-	5
16	-	x^2	-	25
64	-	x^3	-	125
84	-	sums	-	155
100		but should be		100
-16	-	errors	-	$+55$

the sum of which is 71.
Then as 71 : 1 :: 16 : ·2.
Hence $x = 4·2$ nearly.

Again, suppose 4·2 and 4·3, and repeat the work as follows;

1st Sup.				2d Sup.
4·2	-	x	-	4·3
17·64	-	x^2	-	18·49
74·088	-	x^3	-	79·507
95·928		sums		102·297
100				100
$-4·072$		errors		$+2·297$

the sum of which is 6·369.
As 6·369 : 1 :: 2·297 : 0·036
This taken from - 4·300
leaves x nearly = 4·264

Again, suppose 4·264, and 4·265, and work as follows;

4·264	-	x	-	4·265
18·181696	-	x^2	-	18·190225
77·526752	-	x^3	-	77·581310
99·972448	-	sums	-	100·036535
100				100
$-0·027552$	-	errors	-	$+0·036535$

the sum of which is ·064087.
Then as ·064087 : ·001 :: ·027552 : 0·0004299
To this adding - 4·264
gives x very nearly = 4·2644299

The

The work of the example above might have been much shortened, by the use of the Table of Powers in the Arithmetic, which would have given two or three figures by inspection. But the example has been worked out so particularly as it is, the better to show the method.

Ex. 2. To find the root of the equation $x^3 - 15x^2 + 63x = 50$, or the value of x in it.

Here it soon appears that x is very little above 1.

Suppose therefore 1·0 and 1·1, and work as follows:

1·0	x	1·1
63·0	$63x$	69·3
−15	$-15x^2$	−18·15
1	x^3	1·331
49	sums	52·481
50		50
−1	errors	+2·481

3·481 sum of the errors.

As 3·481 : 1 :: ·1 : ·03 correct.

1·00

Hence x = 1·03 nearly.

Again, suppose the two numbers 1·03 and 1·02, &c, as follows:

1·03	x	1·02
64·89	$63x$	64·26
−15·9135	$-15x^2$	−15·6060
1·092727	x^3	1·061208
50·069227	sums	49·715208
50		50
+·069227	errors	−·284792
·284792		

As ·354019 : ·01 :: ·069227 : ·0019555

This taken from 1·03

leaves x nearly = 1·02804

Note 3. Every equation has as many roots as it contains dimensions, or as there are units in the index of its highest power. That is, a simple equation has only one value of the root; but a quadratic equation has two values or roots, a cubic equation has three roots, a biquadratic equation has four roots, and so on.

And when one of the roots of an equation has been found by approximation, as above, the rest may be found as follows. Take, for a dividend, the given equation, with the known term transposed, with its sign changed, to the unknown side of the equation; and, for a divisor, take x minus the root just found. Divide the said dividend by the divisor, and the quotient will be the equation depressed a degree lower than the given one.

Find

Find a root of this new equation by approximation, as before, or otherwise, and it will be a second root of the original equation. Then, by means of this root, depress the second equation one degree lower, and from thence find a third root, and so on, till the equation be reduced to a quadratic; then the two roots of this being found, by the method of completing the square, they will make up the remainder of the roots. Thus, in the foregoing equation, having found one root to be 1·02804, connect it by minus with x for a divisor, and the equation for a dividend, &c, as follows:

$$x - 1{\cdot}02804)\ x^3 - 15x^2 + 63x - 50\ (\ x^2 - 13{\cdot}97196x + 48{\cdot}63627 = 0.$$

Then the two roots of this quadratic equation, or – – – $x^2 - 13{\cdot}97196x = -48{\cdot}63627$, by completing the square, are 6·57653 and 7·39543, which are also the other two roots of the given cubic equation. So that all the three roots of that equation, viz. $x^3 - 15x^2 + 63x = 50$,

are 1·02804
and 6·57653
and 7·39543
sum 15·00000

and the sum of all the roots is found to be 15, being equal to the co-efficient of the 2d term of the equation, which the sum of the roots always ought to be, when they are right.

Note 4. It is also a particular advantage of the foregoing rule, that it is not necessary to prepare the equation, as for other rules, by reducing it to the usual final form and state of equations. Because the rule may be applied at once to an unreduced equation, though it be ever so much embarrassed by surd and compound quantities. As in the following example:

Ex. 3. Let it be required to find the root x of the equation $\sqrt{144x^2 - (x^2 + 20)^2} + \sqrt{196x^2 - (x^2 + 24)^2} = 114$, or the value of x in it.

By a few trials, it is soon found that the value of x is but little above 7. Suppose therefore first that x is $= 7$, and then $x = 8$.

First, when $x = 7$,			Second, when $x = 8$,
47·906	- $\sqrt{144x^2-(x^2+20)^2}$	-	46·476
65·384	- $\sqrt{196x^2-(x^2+24)^2}$	-	69·283
113·290	- the sums of these	-	115·759
114·000	- the true number	-	114·000
−0·710	- the two errors	-	+1·759
+1·759			

As 2·469 : 1 :: 0·710 : 0·2 nearly
7·0

Therefore $x = 7·2$ nearly

Suppose again $x = 7·2$, and then, because it turns out too great, suppose x also $= 7·1$, &c, as follows:

Supp. $x = 7·2$			Supp. $x = 7·1$
47·990	- $\sqrt{144x^2-(x^2+20)^2}$	-	47·973
66·402	- $\sqrt{196x^2-(x^2+24)^2}$	-	65·904
114·392	- the sums of these	-	113·877
114·000	- the true number	-	114·000
+0·392	- the two errors	-	−0·123
0·123			

As ·515 : ·1 :: ·123 : ·024 the correction,
7·100 add

Therefore $x = 7·124$ nearly the root required.

Note 5. The same rule also, among other more difficult forms of equations, succeeds very well in what are called exponential ones, or those which have an unknown quantity in the exponent of the power; as in the following example:

Ex. 4. To find the value of x in the exponential equation $x^x = 100$.

For more easily resolving such kinds of equations, it is convenient to take the logarithms of them, and then compute the terms by means of a table of logarithms. Thus, the logarithms of the two sides of the present equation are

$x \times$ log.

$x \times$ log. of $x = 2$ the log. of 100. Then, by a few trials, it is soon perceived that the value of x is somewhere between the two numbers 3 and 4, and indeed nearly in the middle between them, but rather nearer the latter than the former. Taking therefore first $x = 3{\cdot}5$, and then $= 3{\cdot}6$, and working with the logarithms, the operation will be as follows:

First Supp. $x = 3{\cdot}5$.	Second Supp. $x = 3{\cdot}6$.
Log. of $3{\cdot}5 = 0{\cdot}544068$	Log. of $3{\cdot}6 = 0{\cdot}556303$
then $3{\cdot}5 \times$ log. $3{\cdot}5 = 1{\cdot}904238$	then $3{\cdot}6 \times$ log. $3{\cdot}6 = 2{\cdot}002689$
the true number $2{\cdot}000000$	the true number $2{\cdot}000000$
error, too little, $-{\cdot}095762$	error, too great, $+{\cdot}002689$
002689	

${\cdot}098451$ sum of the errors. Then,

As ${\cdot}098451 : {\cdot}1 :: {\cdot}002689 : 0{\cdot}00273$ the correction
taken from $3{\cdot}60000$

leaves - $3{\cdot}59727 = x$ nearly.

On trial, this is found to be a very small matter too little. Take therefore again, $x = 3{\cdot}59727$, and next $= 3{\cdot}59728$, and repeat the operation as follows:

First, Supp. $x = 3{\cdot}59727$.	Second, Supp. $x = 3{\cdot}59728$.
Log. of $3{\cdot}59727$ is $0{\cdot}555973$	Log. of $3{\cdot}59728$ is $0{\cdot}555974$
$3{\cdot}59727 \times$ log.	$3{\cdot}59728 \times$ log.
of $3{\cdot}59727 = 1{\cdot}9999854$	of $3{\cdot}59728 = 1{\cdot}9999953$
the true number $2{\cdot}0000000$	the true number $2{\cdot}0000000$
error, too little, $-0{\cdot}0000146$	error, too little, $-0{\cdot}0000047$
$-0{\cdot}0000047$	

$0{\cdot}0000099$ diff. of the errors. Then,

As ${\cdot}0000099 : {\cdot}00001 :: {\cdot}0000047 : 0{\cdot}00000474747$ the cor.
added to - $3{\cdot}59728000000$

gives nearly the value of $x = 3{\cdot}59728474747$

Ex. 5. To find the value of x in the equation $x^3 + 10x^2 + 5x = 260$. Ans. $x = 4{\cdot}1179857$.

Ex. 6. To find the value of x in the equation $x^3 - 2x = 50$.
Ans. 3·8648854.

Ex. 7.

Ex. 7. To find the value of x in the equation $x^3 + 2x^2 - 23x = 70$. Ans. $x = 5\cdot13457$.

Ex. 8. To find the value of x in the equation $x^3 - 17x^2 + 54x = 350$. Ans. $x = 14\cdot95407$.

Ex. 9. To find the value of x in the equation $x^4 - 3x^2 - 75x = 10000$. Ans. $x = 10\cdot2609$.

Ex. 10. To find the value of x in the equation $2x^4 - 16x^3 + 40x^2 - 30x = -1$. Ans. $x = 1\cdot284724$.

Ex. 11. To find the value of x in the equation $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x = 54321$. Ans. $x = 8\cdot414455$.

Ex. 12. To find the value of x in the equation $x^x = 123456789$. Ans. $x = 8\cdot6400268$.

Ex. 13. Given $2x^4 - 7x^3 + 11x^2 - 3x = 11$, to find x.

Ex. 14. To find the value of x in the equation

$$(3x^2 - 2\sqrt{x} + 1)^{\frac{3}{5}} - (x^2 - 4x\sqrt{x} + 3\sqrt{x})^{\frac{5}{3}} = 56.$$

Ans. $x = 18\cdot360877$.

To resolve Cubic Equations by Cardan's Rule.

Though the foregoing general method, by the application of Double Position, be the readiest way, in real practice, of finding the roots in numbers of cubic equations, as well as of all the higher equations universally, we may here add the particular method commonly called Cardan's Rule, for resolving cubic equations, in case any person should choose occasionally to employ that method.

The form that a cubic equation must necessarily have, to be resolved by this rule, is this, viz. $z^3 * az = b$, that is, wanting the second term, or the term of the 2d power z^2. Therefore, after any cubic equation has been reduced down to its final usual form, $x^3 + px^2 + qx = r$, freed from the coefficient of its first term, it will then be necessary to take away the 2d term px^2; which is to be done in this manner: Take $\frac{1}{3}p$, or $\frac{1}{3}$ of the coefficient of the second term, and annex it, with the contrary sign, to another unknown letter z, thus $z - \frac{1}{3}p$; then substitute this for x, the unknown letter in the original equation $x^3 + px^2 + qx = r$, and there will result this reduced equation $z^3 * az = b$, of the form proper for applying the following, or Cardan's rule. Or take $c = \frac{1}{3}a$, and $d = \frac{1}{2}b$, by which the reduced equation takes this form, $z^3 * 3cz = 2d$.

Then

Then substitute the values of c and d in this

form, $z = \sqrt[3]{d + \sqrt{(d^2 + c^3)}} + \sqrt[3]{d - \sqrt{(d^2 + c^3)}}$,

or $z = \sqrt[3]{d + \sqrt{(d^2 + c^3)}} - \dfrac{c}{\sqrt[3]{d + \sqrt{(d^2 + c^3)}}}$,

and the value of the root z, of the reduced equation $z^3 * az = b$, will be obtained. Lastly, take $x = z - \frac{1}{3}p$, which will give the value of x, the required root of the original equation $x^3 + px^2 + qx = r$, first proposed.

One root of this equation being thus obtained, then depressing the original equation one degree lower, after the manner described p. 250 and 251, the other two roots of that equation will be obtained by means of the resulting quadratic equation.

Note. When the coefficient a, or c, is negative, and c^3 is greater than d^2, this is called the irreducible case, because then the solution cannot be generally obtained by this rule.

Ex. To find the roots of the equation $x^3 - 6x^2 + 10x = 8$.

First, to take away the 2d term, its coefficient being -6, its 3d part is -2; put therefore $x = z + 2$; then

$$\begin{array}{rcl} x^3 &=& z^3 + 6z^2 + 12z + 8 \\ -\ 6x^2 &=& \quad\ - 6z^2 - 24z - 24 \\ +10x &=& \qquad\qquad + 10z + 20 \\ \hline \text{theref. the sum } z^3 & * & - \ 2z + 4 = 8 \\ \text{or } z^3 & * & - \ 2z = 4 \end{array}$$

Here then $a = -2$, $b = 4$, $c = -\frac{2}{3}$, $d = 2$.

Theref. $\sqrt[3]{d + \sqrt{(d^2 + c^3)}} = \sqrt[3]{2 + \sqrt{(4 - \frac{8}{27})}} = \sqrt[3]{2 + \sqrt{\frac{100}{27}}} =$
$\sqrt[3]{2 + \frac{10}{9}\sqrt{3}} = 1{\cdot}57735$

and $\sqrt[3]{d - \sqrt{(d^2 + c^3)}} = \sqrt[3]{2 - \sqrt{(4 - \frac{8}{27})}} = \sqrt[3]{2 - \sqrt{\frac{100}{27}}} =$
$\sqrt[3]{2 - \frac{10}{9}\sqrt{3}} = 0{\cdot}42265$

then the sum of these two is the value of $z = 2$.

Hence $x = z + 2 = 4$, one root of x in the eq. $x^3 - 6x^2 + 10x = 8$.

To find the two other roots, perform the division, &c. as in p. 251, thus:

$$\begin{array}{r} x - 4\ )\ x^3 - 6x^2 + 10x - 8\ (\ x^2 - 2x + 2 = 0 \\ x^3 - 4x^2 \qquad\qquad\qquad\qquad\quad \\ \hline -2x^2 + 10x \qquad\qquad\qquad \\ -2x^2 + \ \ 8x \qquad\qquad\qquad \\ \hline 2x - 8 \qquad\qquad\quad \\ 2x - 8 \qquad\qquad\quad \\ \hline \end{array}$$

Hence

Hence $x^2-2x=-2$, or $x^2-2x+1=-1$, and $x-1=\pm\sqrt{-1}$; $x=1+\sqrt{-1}$ or $=1-\sqrt{-1}$, the two other sought.

Ex. 2. To find the roots of $x^3-9x^2+28x=30$.

Ans. $x=3$, or $=3+\sqrt{-1}$, or $=3-\sqrt{-1}$.

Ex. 3. To find the roots of $x^3-7x^2+14x=20$.

Ans. $x=5$, or $=1+\sqrt{-3}$, or $=1-\sqrt{-3}$.

OF SIMPLE INTEREST.

As the interest of any sum, for any time, is directly proportional to the principal sum, and to the time; therefore the interest of 1 pound, for 1 year, being multiplied by any given principal sum, and by the time of its forbearance, in years and parts, will give its interest for that time. That is, if there be put

$r=$ the rate of interest of 1 pound per annum,
$p=$ any principal sum lent,
$t=$ the time it is lent for, and
$a=$ the amount or sum of principal and interest; then is $prt=$ the interest of the sum p, for the time t, and conseq. $p+prt$ or $p\times(1+rt)=a$, the amount for that time.

From this expression, other theorems can easily be deduced, for finding any of the quantities above mentioned: which theorems, collected together, will be as below:

1st, $a=p+prt$, the amount,

2d, $p=\dfrac{a}{1+rt}$, the principal,

3d, $r=\dfrac{a-p}{pt}$, the rate,

4th, $t=\dfrac{a-p}{pr}$, the time.

For Example. Let it be required to find, in what time any principal sum will double itself, at any rate of simple interest.

In this case, we must use the first theorem, $a=p+prt$, in which the amount a must be made $=2p$, or double the principal, that is, $p+prt=2p$, or $prt=p$, or $rt=1$; and hence $t=\dfrac{1}{r}$.

Here,

Here, r being the interest of 1*l.* for 1 year, it follows, that the doubling at simple interest, is equal to the quotient of any sum divided by its interest for 1 year. So, if the rate of interest be 5 per cent. then $100 \div 5 = 20$, is the time of doubling at that rate.

Or the 4th theorem gives at once

$$t = \frac{a-p}{pr} = \frac{2p-p}{pr} = \frac{2-1}{r} = \frac{1}{r}, \text{ the same as before.}$$

COMPOUND INTEREST.

Besides the quantities concerned in Simple Interest, namely,

p = the principal sum,
r = the rate or interest of 1*l.* for 1 year,
a = the whole amount of the principal and interest,
t = the time,

there is another quantity employed in Compound Interest, viz. the ratio of the rate of interest, which is the amount of 1*l.* for 1 time of payment, and which here let be denoted by R, viz.

$R = 1 + r$, the amount of 1*l.* for 1 time.

Then the particular amounts for the several times may be thus computed, viz. As 1*l.* is to its amount for any time, so is any proposed principal sum, to its amount for the same time; that is, as

1*l.* : R : : p : pR, the 1st year's amount,
1*l.* : R : : pR : $p\mathrm{R}^2$, the 2d year's amount,
1*l.* : R : : $p\mathrm{R}^2$: $p\mathrm{R}^3$, the 3d year's amount,
and so on.

Therefore, in general, $p\mathrm{R}^t = a$ is the amount for the t year, or t time of payment. Whence the following general theorems are deduced:

1st, $a = p\mathrm{R}^t$, the amount,

2d, $p = \frac{a}{\mathrm{R}^t}$ the principal,

3d, $\mathrm{R} = \sqrt[t]{\frac{a}{p}}$, the ratio,

4th, $t = \frac{\text{log. of } a - \text{log. of } p}{\text{log. of R}}$, the time.

From which, any one of the quantities may be found, when the rest are given.

As to the whole interest, it is found by barely subtracting the principal p from the amount a.

Example. Suppose it be required to find, in how many years any principal sum will double itself, at any proposed rate of compound interest.

In this case the 4th theorem must be employed, making $a = 2p$; and then it is

$$t = \frac{\log. a - \log. p}{\log. R.} = \frac{\log. 2p - \log. p}{\log. R.} = \frac{\log. 2}{\log. R}.$$

So, if the rate of interest be 5 per cent. per annum; then $R = 1 + \cdot 05 = 1\cdot 05$; and hence

$$t = \frac{\log. 2}{\log. 1\cdot 05} = \frac{\cdot 301030}{\cdot 021189} = 14\cdot 2067 \text{ nearly};$$

that is, any sum doubles itself in $14\frac{1}{5}$ years nearly, at the rate of 5 per cent. per annum compound interest.

Hence, and from the like question in Simple Interest, above given, are deduced the times in which any sum doubles itself, at several rates of interest, both simple and compound; viz.

At		At Simp. Int.	At Comp. Int.
2	per cent. per annum interest, 1*l*. or any other sum, will double itself in the following years.	in 50	in 35·0028
$2\frac{1}{2}$		40	28·0701
3		$33\frac{1}{3}$	23·4498
$3\frac{1}{2}$		$28\frac{4}{7}$	20·1488
4		25	17·6730
$4\frac{1}{2}$		$22\frac{2}{9}$	15·7473
5		20 Years.	14·2067 Years.
6		$16\frac{2}{3}$	11·8957
7		$14\frac{2}{7}$	10·2448
8		$12\frac{1}{2}$	9·0065
9		$11\frac{1}{9}$	8·0432
10		10	7·2725

The

The following Table will very much facilitate calculations of compound interest on any sum, for any number of years, at various rates of interest.

The Amounts of 1*l.* in any Number of Years.

Yrs.	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	6
1	1·0300	1·0350	1·0400	1·0450	1·0500	1·0600
2	1·0609	1·0712	1·0816	1·0920	1·1025	1·1236
3	1·0927	1·1087	1·1249	1·1412	1·1576	1·1910
4	1·1255	1·1475	1·1699	1·1925	1·2155	1·2625
5	1·1593	1·1877	1·2167	1·2462	1·2763	1·3382
6	1·1941	1·2293	1·2653	1·3023	1·3401	1·4185
7	1·2299	1·2723	1·3159	1·3609	1·4071	1·5036
8	1·2668	1·3168	1·3686	1·4221	1·4775	1·5939
9	1·3048	1·3629	1·4233	1·4861	1·5513	1·6895
10	1·3439	1·4106	1·4802	1·5530	1·6289	1·7909
11	1·3842	1·4600	1·5895	1·6229	1·7103	1·8983
12	1·4258	1·5111	1·6010	1·6959	1·7959	2·0122
13	1·4685	1·5640	1·6651	1·7722	1·8856	2·1329
14	1·5126	1·6187	1·7317	1·8519	1·9799	2·2609
15	1·5580	1·6753	1·8009	1·9353	2·0789	2·3966
16	1·6047	1·7340	1·8730	2·0224	2·1829	2·5404
17	1·6528	1·7947	1·9479	2·1134	2·2920	2·6928
18	1·7024	1·8575	2·0258	2·2085	2·4066	2·8543
19	1·7535	1·9225	2·1068	2·3079	2·5270	3·0256
20	1·8061	1·9898	2·1911	2·4117	2·6533	3·2071

The use of this Table, which contains all the powers, R^t, to the 20th power, or the amounts of 1*l*, is chiefly to calculate the interest, or the amount of any principal sum, for any time, not more than 20 years.

For example, let it be required to find, to how much 523*l.* will amount in 15 years, at the rate of 5 per cent per annum compound interest.

In the table, on the line 15, and in the column 5 per cent,

is the amount of 1*l*, viz.	- -	2·0789
this multiplied by the principal	-	523
gives the amount	- -	1087·2647
or	- -	1087*l.* 5*s.* $3\frac{1}{4}$*d.*
and therefore the interest is		564*l.* 5*s.* $3\frac{1}{4}$*d.*

Note 1. When the rate of interest is to be determined to any other time than a year; as suppose to $\frac{1}{2}$ a year, or $\frac{1}{4}$ a year, &c; the rules are still the same; but then t will

express that time, and R must be taken the amount for that time also.

Note 2. When the compound interest, or amount, of any sum, is required for the parts of a year; it may be determined in the following manner:

1*st*, For any time which is some aliquot part of a year:— Find the amount of 1*l.* for 1 year, as before; then that root of it which is denoted by the aliquot part, will be the amount of 1*l.* This amount being multiplied by the principal sum, will produce the amount of the given sum as required.

2*d*, When the time is not an aliquot part of a year:— Reduce the time into days, and take the 365th root of the amount of 1*l.* for 1 year, which will give the amount of the same for 1 day. Then raise this amount to that power whose index is equal to the number of days, and it will be the amount for that time. Which amount being multiplied by the principal sum, will produce the amount of that sum as before.—And in these calculations, the operation by logarithms will be very useful.

OF ANNUITIES.

ANNUITY is a term used for any periodical income, arising from money lent, or from houses, lands, salaries, pensions, &c. payable from time to time, but mostly by annual payments.

Annuities are divided into those that are in Possession, and those in Reversion: the former meaning such as have commenced; and the latter such as will not begin till some particular event has happened, or till after some certain time has elapsed.

When an annuity is forborn for some years, or the payments not made for that time, the annuity is said to be in Arrears.

An annuity may also be for a certain number of years; or it may be without any limit, and then it is called a Perpetuity.

The Amount of an annuity, forborn for any number of years, is the sum arising from the addition of all the annuities for that number of years, together with the interest due upon each after it becomes due.

The

The Present Worth or Value of an annuity, is the price or sum which ought to be given for it, supposing it to be bought off, or paid all at once.

Let a = the annuity, pension, or yearly rent;
n = the number of years forborn, or lent for;
R = the amount of $1l.$ for 1 year;
m = the amount of the annuity;
v = its value, or its present worth.

Now, 1 being the present value of the sum R, by proportion the present value of any other sum a, is thus found: as $R : 1 :: a : \frac{a}{R}$ the present value of a due 1 year hence. In like manner $\frac{a}{R^2}$ is the present value of a due 2 years hence; for $R : 1 :: \frac{a}{R} : \frac{a}{R^2}$. So also $\frac{a}{R^3}$, $\frac{a}{R^4}$, $\frac{a}{R^5}$, &c, will be the present values of a, due at the end of 3, 4, 5, &c, years respectively. Consequently the sum of all these, or $\frac{a}{R} + \frac{a}{R^2} + \frac{a}{R^3} + \frac{a}{R^4} + \&c = (\frac{1}{R} + \frac{1}{R^2} + \frac{1}{R^3} + \frac{1}{R^4}\ \&c.) \times a$, continued to n terms, will be the present value of all the n years' annuities. And the value of the perpetuity, is the sum of the series to infinity.

But this series, it is evident, is a geometrical progression, having $\frac{1}{R}$ both for its first term and common ratio, and the number of its terms n; therefore the sum v of all the terms, or the present value of all the annual payments, will be

$$v = \frac{\frac{1}{R} - \frac{1}{R} \times \frac{1}{R^n}}{1 - \frac{1}{R}} \times a, \text{ or } = \frac{R^n - 1}{R - 1} \times \frac{a}{R^n}.$$

When the annuity is a perpetuity; n being infinite, R^n is also infinite, and therefore the quantity $\frac{1}{R^n}$ becomes $= 0$, therefore $\frac{a}{R-1} \times \frac{1}{R^n}$ also $= 0$; consequently the expression becomes barely $v = \frac{a}{R-1}$; that is, any annuity divided by the interest of $1l.$ for 1 year, gives the value of the perpetuity. So, if the rate of interest be 5 per cent,

Then $100a \div 5 = 20a$ is the value of the perpetuity at 5 per cent: Also $100a \div 4 = 25a$ is the value of the perpetuity

petuity at 4 per cent: And $100a \div 3 = 33\frac{1}{3}a$ is the value of the perpetuity at 3 per cent: and so on.

Again, because the amount of 1*l.* in n years, is R^n, its increase in that time will be $R^n - 1$; but its interest for one single year, or the annuity answering to that increase, is $R - 1$; therefore as $R - 1$ is to $R^n - 1$, so is a to m; that is, $m = \frac{R^n - 1}{R - 1} \times a$. Hence, the several cases relating to Annuities in Arrear, will be resolved by the following equations:

$$m = \frac{R^n - 1}{R - 1} \times a = vR^n;$$

$$v = \frac{R^n - 1}{R - 1} \times \frac{a}{R^n} = \frac{m}{R^n};$$

$$a = \frac{R - 1}{R^n - 1} \times m = \frac{R - 1}{R^n - 1} \times vR^n;$$

$$n = \frac{\log. m - \log. v}{\log. R} = \frac{\log.\frac{mR - m + a}{a}}{\log. R};$$

$$\text{Log. } R = \frac{\log. m - \log. v}{n};$$

$$r = \left(\frac{1}{R^p} - \frac{1}{R^n}\right) \times \frac{a}{R - 1}.$$

In this last theorem, r denotes the present value of an annuity in reversion, after p years, or not commencing till after the first p years, being found by taking the difference between the two values $\frac{R^n - 1}{R - 1} \times \frac{a}{R^n}$ and $\frac{R^p - 1}{R - 1} \times \frac{a}{R^p}$, for n years and p years.

But the amount and present value of any annuity for any number of years, up to 21, will be most readily found by the two following tables.

TABLE I.

The Amount of an Annuity of 1*l.* at Compound Interest.

Yrs.	at 3 per c.	3½ per c.	4 per c.	4½ per c.	5 per c.	6 per c.
1	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
2	2·0300	2·0350	2·0400	2·0450	2·0500	2·0600
3	3·0909	3·1062	3·1216	3·1370	3·1525	3·1836
4	4·1836	4·2149	4·2465	4·2782	4·3101	4·3746
5	5·3091	5·3625	5·4163	5·4707	5·5256	5·6371
6	6·4684	6·5502	6·6330	6·7169	6·8019	6·9753
7	7·6625	7·7794	7·8983	8·0192	8·1420	8·3938
8	8·8923	9·0517	9·2142	9·3800	9·5491	9·8975
9	10·1591	10·3685	10·5828	10·8021	11·0266	11·4913
10	11·4639	11·7314	12·0061	12·2882	12·5779	13·1808
11	12·8078	13·1420	13·4864	13·8412	14·2068	14·9716
12	14·1920	14·6020	15·0258	15·4640	15·9171	16·8699
13	15·6178	16·1130	16·6268	17·1599	17·7130	18·8821
14	17·0863	17·6770	18·2919	18·9321	19·5986	21·0151
15	18·5989	19·2957	20·3236	20·7841	21·5786	23·2760
16	20·1569	20·9710	21·8245	22·7193	23·6575	25·6725
17	21·7616	22·7050	23·6975	24·7417	25·8404	28·2129
18	23·4144	24·4997	25·6454	26·8551	28·1324	30·9057
19	25·1169	26·3572	27·6712	29·0636	30·5390	33·7600
20	26·8704	28·2797	29·7781	31·3714	33·0660	36·7856
21	28·6765	30·2695	31·9692	33·7831	35·7193	39·9927

TABLE II. The Present Value of an Annuity of 1*l.*

Yrs.	at 3 per c.	3½ per c.	4 per c.	4½ per c.	5 per c.	6 per c.
1	0·9709	0·9662	0·9615	0·9569	0·9524	0·9434
2	1·9135	1·8997	1·8861	1·8727	1·8594	1·8334
3	2·8286	2·8016	2·7751	2·7490	2·7233	2·6730
4	3·7171	3·6731	3·6299	3·5875	3·5460	3·4651
5	4·5797	4·5151	4·4518	4·3900	4·3295	4·2124
6	5·4172	5·3286	5·2421	5·1579	5·0757	4·9173
7	6·2303	6·1145	6·0020	5·8927	5·7864	5·5824
8	7·0197	6·8740	6·7327	6·5959	6·4632	6·2098
9	7·7861	7·6077	7·4353	7·2688	7·1078	6·8017
10	8·5302	8·3166	8·1109	7·9127	7·7217	7·3601
11	9·2526	9·0116	8·7605	8·5289	8·3054	7·8869
12	9·9540	9·6633	9·3851	9·1186	8·8633	8·3838
13	10·6350	10·3027	9·9857	9·6829	9·3936	8·8527
14	11·2961	10·9205	10·5631	10·2228	9·8986	9·2950
15	11·9379	11·5174	11·1184	10·7396	10·3797	9·7123
16	12·5611	12·0941	11·6523	11·2340	10·8378	10·1059
17	13·1661	12·6513	12·1657	11·7072	11·2741	10·4773
18	13·7535	13·1897	12·6593	12·1600	11·6896	10·8276
19	14·3238	13·7098	13·1339	12·5933	12·0853	11·1581
20	14·8775	14·2124	13·5903	13·0079	12·4622	11·4699
21	15·4150	14·6980	14·0292	13·4047	12·8212	11·7641

To

To find the Amount of any annuity forborn a certain number of years.

TAKE out the amount of 1*l.* from the first table, for the proposed rate and time; then multiply it by the given annuity; and the product will be the amount, for the same number of years, and rate of interest.—And the converse to find the rate or time.

Exam. To find how much an annuity of 50*l.* will amount to in 20 years, at $3\frac{1}{2}$ per cent. compound interest.

On the line of 20 years, and in the column of $3\frac{1}{2}$ per cent. stands 28·2797, which is the amount of an annuity of 1*l.* for the 20 years. Then $28{\cdot}2797 \times 50$, gives 1413·985*l.* = 1413*l.* 19*s.* 8*d.* for the answer required.

To find the Present Value of any annuity for any number of years.—Proceed here by the 2d table, in the same manner as above for the 1st table, and the present worth required will be found.

Exam. 1. To find the present value of an annuity of 50*l.* which is to continue 20 years, at $3\frac{1}{2}$ per cent.—By the table, the present value of 1*l.* for the given rate and time, is 14·2124; therefore $14{\cdot}2124 \times 50 = 710{\cdot}62$*l.* or 710*l.* 12*s.* 4*d.* is the present value required.

Exam. 2. To find the present value of an annuity of 20*l.* to commence 10 years hence, and then to continue for 11 years longer, or to terminate 21 years hence, at 4 per cent. interest.—In such cases as this, we have to find the difference between the present values of two equal annuities, for the two given times; which therefore will be done by subtracting the tabular value of the one period from that of the other, and then multiplying by the given annuity. Thus,

tabular value for 21 years	14·0292
ditto for - 10 years	8·1109
the difference	5·9183
multiplied by	20
gives -	118·366*l.*
or - -	118*l.* 7*s.* $3\frac{1}{2}$*d.* the answer.

END OF THE ALGEBRA.

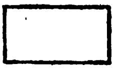
GEOMETRY.

DEFINITIONS.

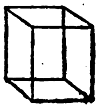
1. A POINT is that which has position, but no magnitude, nor dimensions; neither length, breadth, nor thickness.

2. A Line is length, without breadth or thickness.

3. A Surface or Superficies, is an extension or a figure of two dimensions, length and breadth; but without thickness.

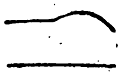


4. A Body or Solid, is a figure of three dimensions, namely, length, breadth, and depth, or thickness.



5. Lines are either Right, or Curved, or Mixed of these two.

6. A Right Line, or Straight Line, lies all in the same direction, between its extremities; and is the shortest distance between two points.

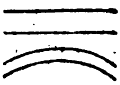


When a Line is mentioned simply, it means a Right Line.

7. A Curve continually changes its direction between its extreme points.

8. Lines are either Parallel, Oblique, Perpendicular, or Tangential.

9. Parallel Lines are always at the same perpendicular distance; and they never meet, though ever so far produced.



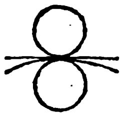
10. Oblique lines change their distance, and would meet, if produced on the side of the least distance.

11. One line is Perpendicular to another, when it inclines not more on the one side

than

than the other, or when the angles on both sides of it are equal.

12. A line or circle is Tangential, or a Tangent to a circle, or other curve, when it touches it, without cutting, when both are produced.



13. An Angle is the inclination or opening of two lines, having different directions, and meeting in a point.

14. Angles are Right or Oblique, Acute or Obtuse.

15. A Right Angle is that which is made by one line perpendicular to another. Or when the angles on each side are equal to one another, they are right angles.

16. An Oblique Angle is that which is made by two oblique lines; and is either less or greater than a right angle.

17. An Acute Angle is less than a right angle.

18. An Obtuse Angle is greater than a right angle.

19. Superficies are either Plane or Curved.

20. A Plane Superficies, or a Plane, is that with which a right line may, every way, coincide. Or, if the line touch the plane in two points, it will touch it in every point. But, if not, it is curved.

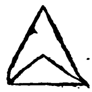
21. Plane Figures are bounded either by right lines or curves.

22. Plane figures that are bounded by right lines have names according to the number of their sides, or of their angles; for they have as many sides as angles; the least number being three.

23. A figure of three sides and angles is called a Triangle. And it receives particular denominations from the relations of its sides and angles.

24. An Equilateral Triangle is that whose three sides are all equal.

25. An Isosceles Triangle is that which has two sides equal.



26. A

26. A Scalene Triangle is that whose three sides are all unequal.

27. A Right-angled Triangle is that which has one right-angle.

28. Other triangles are Oblique-angled, and are either Obtuse or Acute.

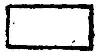
29. An Obtuse-angled Triangle has one obtuse angle.

30. An Acute-angled Triangle has all its three angles acute.

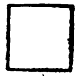
31. A figure of Four sides and angles is called a Quadrangle, or a Quadrilateral.

32. A Parallelogram is a quadrilateral which has both its pairs of opposite sides parallel. And it takes the following particular names, viz. Rectangle, Square, Rhombus, Rhomboid.

33. A Rectangle is a parallelogram having a right angle.



34. A Square is an equilateral rectangle; having its length and breadth equal.



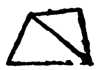
35. A Rhomboid is an oblique-angled parallelogram.

36. A Rhombus is an equilateral rhomboid; having all its sides equal, but its angles oblique.

37. A Trapezium is a quadrilateral which hath not its opposite sides parallel.

38. A Trapezoid has only one pair of opposite sides parallel.

39. A Diagonal is a line joining any two opposite angles of a quadrilateral.



40. Plane figures that have more than four sides are, in general, called Polygons: and they receive other particular names, according to the number of their sides or angles. Thus,

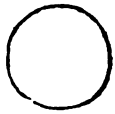
41. A Pentagon is a polygon of five sides; a Hexagon, of six sides; a Heptagon, seven; an Octagon, eight; a N- nagon, nine; a Decagon, ten; an Undecag Dodecagon, twelve sides,

42. A Regular Polygon has all its sides and all its angles equol.—If they are not both equal, the polygon is Irregular.

43. An Equilateral Triangle is also a Regular Figure of three sides, and the Square is one of four; the former being also called a Trigon, and the latter a Tetragon.

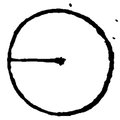
44. Any figure is equilateral, when all its sides are equal: and it is equiangular when all its angles are equal. When both these are equal, it is a regular figure.

45. A Circle is a plane figure bounded by a curve line, called the Circumference, which is every where equidistant from a certain point within, called its Centre.

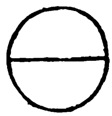


The circumference itself is often called a circle, and also the Periphery.

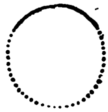
46. The Radius of a circle is a line drawn from the centre to the circumference.



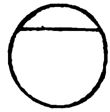
47. The Diameter of a circle is a line drawn through the centre, and terminating at the circumference on both sides.



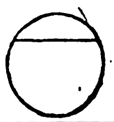
48. An Arc of a circle is any part of the circumference.



49. A Chord is a right line joining the extremities of an arc.



50. A Segment is any part of a circle bounded by an arc and its chord.



51. A Semicircle is half the circle, or a segment cut off by a diameter.

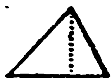
The half circumference is sometimes called the Semicircle.

52. A Sector is any part of a circle which is bounded by an arc, and two radii drawn to its extremities.

53. A Quadrant, or Quarter of a circle, is a sector having a quarter of [illegible]rcumferen[illegible] for its arc, and its two rad[illegible]rpendi[illegible] to each other. A quarter [illegible]umfere[illegible] is sometimes called a Qua[illegible]

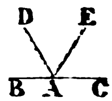
The

54. The Height or Altitude of a figure is a perpendicular let fall from an angle, or its vertex, to the opposite side, called the base.



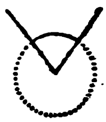
55. In a right-angled triangle, the side opposite the right angle is called the Hypothenuse; and the other two sides are called the Legs, and sometimes the Base and Perpendicular.

56. When an angle is denoted by three letters, of which one stands at the angular point, and the other two on the two sides, that which stands at the angular point is read in the middle.

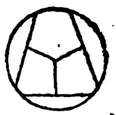


57. The circumference of every circle is supposed to be divided into 360 equal parts, called Degrees: and each degree into 60 Minutes, each minute into 60 Seconds, and so on. Hence a semicircle contains 180 degrees, and a quadrant 90 degrees.

58. The Measure of an angle, is an arc of any circle contained between the two lines which form that angle, the angular point being the centre; and it is estimated by the number of degrees contained in that arc.

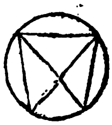


59. Lines, or chords, are said to be Equidistant from the centre of a circle, when perpendiculars drawn to them from the centre are equal.



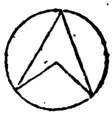
60. And the right line on which the Greater Perpendicular falls, is said to be farther from the centre.

61. An Angle In a segment is that which is contained by two lines, drawn from any point in the arc of the segment, to the two extremities of that arc.



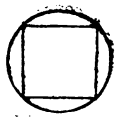
62. An Angle On a segment, or an arc, is that which is contained by two lines, drawn from any point in the opposite or supplemental part of the circumference, to the extremities of the arc, and containing the arc between them.

63. An angle at the circumference, is that whose angular point is any where in the circumference. And an angle at the centre, is that whose angular point is at the centre.

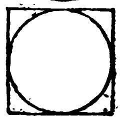


64. A

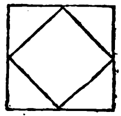
64. A right-lined figure is Inscribed in a circle, or the circle Circumscribes it, when all the angular points of the figure are in the circumference of the circle.



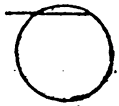
65. A right-lined figure Circumscribes a circle, or the circle is Inscribed in it, when all the sides of the figure touch the circumference of the circle.



66. One right-lined figure is Inscribed in another, or the latter Circumscribes the former, when all the angular points of the former are placed in the sides of the latter.



67. A Secant is a line that cuts a circle, lying partly within, and partly without it.



68. Two triangles, or other right-lined figures, are said to be mutually equilateral, when all the sides of the one are equal to the corresponding sides of the other, each to each: and they are said to be mutually equiangular, when the angles of the one are respectively equal to those of the other.

69. Identical figures, are such as are both mutually equilateral and equiangular; or that have all the sides and all the angles of the one, respectively equal to all the sides and all the angles of the other, each to each; so that if the one figure were applied to, or laid upon the other, all the sides of the one would exactly fall upon and cover all the sides of the other; the two becoming as it were but one and the same figure.

70. Similar figures, are those that have all the angles of the one equal to all the angles of the other, each to each, and the sides about the equal angles proportional.

71. The Perimeter of a figure, is the sum of all its sides taken together.

72. A Proposition, is something which is either proposed to be done, or to be demonstrated, and is either a problem or a theorem.

73. A Problem is something proposed to be done.

74. A Theorem, is something proposed to be demonstrated.

75. A Lemma, is something which is premised, or demonstrated, in order to render what follows more easy.

76. A Corollary, is a consequent truth, gained immediately from some preceding truth, or demonstration.

77. A Scholium, is a remark or observation made upon *something* going before it.

AXIOMS.

AXIOMS.

1. THINGS which are equal to the same thing are equal to each other.

2. When equals are added to equals, the wholes are equal.

3. When equals are taken from equals, the remains are equal.

4. When equals are added to unequals, the wholes are unequal.

5. When equals are taken from unequals, the remains are unequal.

6. Things which are double of the same thing, or equal things, are equal to each other.

7. Things which are halves of the same thing, are equal.

8. Every whole is equal to all its parts taken together.

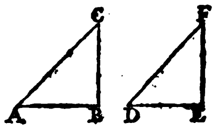
9. Things which coincide, or fill the same space, are identical, or mutually equal in all their parts.

10. All right angles are equal to one another.

11. Angles that have equal measures, or arcs, are equal.

THEOREM I.

IF two Triangles have Two Sides and the Included Angle in the one, equal to Two Sides and the Included Angle in the other, the Triangles will be Identical, or equal in all respects.



In the two triangles ABC, DEF, if the side AC be equal to the side DF, and the side BC equal to the side EF, and the angle C equal to the angle F; then will the two triangles be identical, or equal in all respects.

For conceive the triangle ABC to be applied to, or placed on, the triangle DEF, in such a manner that the point C may

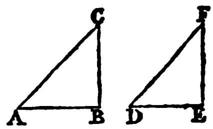
coincide

coincide with the point F, and the side AC with the side DF, which is equal to it.

Then, since the angle F is equal to the angle C (by hyp.), the side BC will fall on the side EF. Also, because AC is equal to DF, and BC equal to EF (by hyp.), the point A will coincide with the point D, and the point B with the point E; consequently the side AB will coincide with the side DE. Therefore the two triangles are identical, and have all their other corresponding parts equal (ax. 9), namely, the side AB equal to the side DE, the angle A to the angle D, and the angle B to the angle E. Q. E. D.

THEOREM II.

WHEN Two Triangles have Two Angles and the included Side in the one, equal to Two Angles and the included Side in the other, the Triangles are Identical, or have their other sides and angle equal.

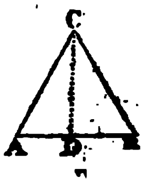


Let the two triangles ABC, DEF, have the angle A equal to the angle D, the angle B equal to the angle E, and the side AB equal to the side DE; then these two triangles will be identical.

For, conceive the triangle ABC to be placed on the triangle DEF, in such manner that the side AB may fall exactly on the equal side DE. Then, since the angle A is equal to the angle D (by hyp.), the side AC must fall on the side DF; and, in like manner, because the angle B is equal to the angle E, the side BC must fall on the side EF. Thus the three sides of the triangle ABC will be exactly placed on the three sides of the triangle DEF: consequently the two triangles are identical (ax. 9), having the other two sides AC, BC, equal to the two DF, EF, and the remaining angle C equal to the remaining angle F. Q. E. D.

THEOREM III.

IN an Isosceles triangle, the Angles at the Base are equal. Or, if a Triangle have Two Sides equal, their Opposite Angles will also be equal.



If the triangle ABC have the side AC equal to the side BC: then will the angle B be equal to the angle A.

For, conceive the angle C to be bisected, or divided into two equal parts, by the line CD, making the angle ACD equal to the angle BCD.

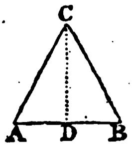
Then, the two triangles ACD, BCD, have two sides and the contained angle of the one, equal to two sides and the contained angle of the other, viz. the side AC equal to BC, the angle ACD equal to BCD, and the side CD common; therefore these two triangles are identical, or equal in all respects (th. 1); and consequently the angle A equal to the angle B. Q. E. D.

Corol. 1. Hence the line which bisects the verticle angle of an isosceles triangle, bisects the base, and is also perpendicular to it.

Corol. 2. Hence too it appears, that every equilateral triangle, is also equiangular, or has all its angles equal.

THEOREM IV.

WHEN a Triangle has Two of its Angles equal, the Sides Opposite to them are also equal.



If the triangle ABC, have the angle A equal to the angle B, it will also have the side AC equal to the side BC.

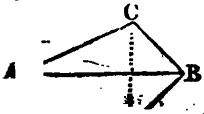
For, conceive the side AB to be bisected in the point D, making AD equal to DB; and join DC, dividing the whole triangle into the two triangles ACD, BCD. Also conceive the triangle ACD to be turned over upon the triangle BCD, so that AD may fall on BD.

Then, because the line AD is equal to the line DB (by hyp.), the point A coincides with the point B, and the point D with the point D. Also, because the angle A is equal to the angle B (by hyp.), the line AC will fall on the line BC, and the extremity C of the side AC will coincide with the extremity C of the side BC, because DC is common to both; consequently the side AC is equal to BC. Q. E. D.

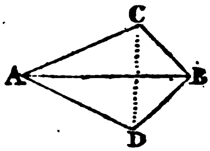
Corol. Hence every equiangular triangle is also equilateral.

THEOREM V.

WHEN Two Triangles have all the Three Sides in the one, equal to all the Three Sides in the other, the Triangles are Identical, or have also their Three Angles equal, each to each.



Let the two triangles ABC, ABD, have their three sides respectively equal, viz. the side AB equal t[illegible] AC to AD, and B[illegible] the two triang[illegible] their angles [illegible]



that are opposite to the equal sides; namely, the angle BAC to the angle BAD, the angle ABC to the angle ABD, and the angle C to the angle D.

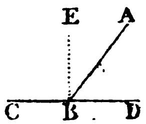
For, conceive the two triangles to be joined together by their longest equal sides, and draw the line CD.

Then, in the triangle ACD, because the side AC is equal to AD (by hyp.), the angle ACD is equal to the angle ADC (th. 3). In like manner, in the triangle BCD, the angle BCD is equal to the angle BDC, because the side BC is equal to BD. Hence then, the angle ACD being equal to the angle ADC, and the angle BCD to the angle BDC, by equal additions the sum of the two angles ACD, BCD, is equal to the sum of the two ADC, BDC, (ax. 2), that is, the whole angle ACB equal to the whole angle ADB.

Since then, the two sides AC, CB, are equal to the two sides AD, DB, each to each, (by hyp.), and their contained angles ACB, ADB, also equal, the two triangles ABC, ABD, are identical (th. 1), and have the other angles equal, viz. the angle BAC to the angle BAD, and the angle ABC to the angle ABD. Q. E. D.

THEOREM VI.

WHEN one Line meets another, the Angles which it makes on the Same Side of the other, are together equal to Two Right Angles.



Let the line AB meet the line CD: then will the two angles ABC, ABD, taken together, be equal to two right angles.

For, first, when the two angles ABC, ABD, are equal to each other, they are both of them right angles (def. 15).

But when the angles are unequal, suppose BE drawn perpendicular to CD. Then, since the two angles EBC, EBD, are right angles (def. 15), and the angle EBD is equal to the two angles EBA, ABD, together (ax. 8), the three angles, EBC, EBA, and ABD, are equal to two right angles.

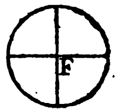
But the two angles EBC, EBA, are together equal to the angle ABC (ax. 8). Consequently the two angles ABC, ABD, are also equal to two right angles. Q. E. D.

Corol. 1. Hence also, conversely, if the two angles ABC, ABD, on both sides of the line AB, make up together two right angles, then CB and BD form one continued right line CD.

Corol.

Corol. 2. Hence, all the angles which can be made, at any point B, by any number of lines, on the same side of the right line CD, are, when taken all together, equal to two right angles.

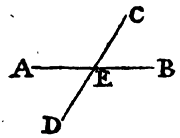
Corol. 3. And, as all the angles that can be made on the other side of the line CD are also equal to two right angles; therefore all the angles that can be made quite round a point B, by any number of lines, are equal to four right angles.



Corol. 4. Hence also the whole circumference of a circle, being the sum of the measures of all the angles that can be made about the centre F (def. 57), is the measure of four right angles. Consequently, a semicircle, or 180 degrees, is the measure of two right angles; and a quadrant, or 90 degrees, the measure of one right angle.

THEOREM VII.

WHEN two Lines Intersect each other, the Opposite Angles are equal.



Let the two lines AB, CD, intersect in the point E; then will the angle AEC be equal to the angle BED; and the angle AED equal to the angle CEB.

For, since the line CE meets the line AB, the two angles AEC, BEC, taken together, are equal to two right angles (th. 6).

In like manner, the line BE, meeting the line CD, makes the two angles BEC, BED, equal to two right angles.

Therefore the sum of the two angles AEC, BEC, is equal to the sum of the two BEC, BED (ax. 1).

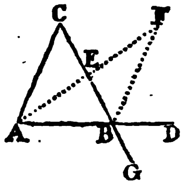
And if the angle BEC, which is common, be taken away from both these, the remaining angle AEC will be equal to the remaining angle BED (ax. 3).

And in like manner it may be shown, that the angle AED is equal to the opposite angle BEC.

THEOREM VIII.

WHEN One Side of a Triangle is produced, the Outward Angle is Greater [illegible] two Inward Opposite Angles.

Let



Let ABC be a triangle, having the side AB produced to D; then will the outward angle CBD be greater than either of the inward opposite angles A or C.

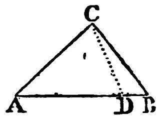
For, conceive the side BC to be bisected in the point E, and draw the line AE, producing it till EF be equal to AE; and join BF.

Then, since the two triangles AEC, BEF, have the side AE = the side EF, and the side CE = the side BE (by suppos.) and the included or opposite angles at E also equal (th. 7), therefore those two triangles are equal in all respects (th. 1), and have the angle C = the corresponding angle EBF. But the angle CBD is greater than the angle EBF; consequently the said outward angle CBD is also greater than the angle C.

In like manner, if CB be produced to G, and AB be bisected, it may be shown that the outward angle ABG, or its equal CBD, is greater than the other angle A.

THEOREM IX.

The Greater Side, of every Triangle, is opposite to the Greater Angle; and the Greater Angle opposite to the Greater Side.



Let ABC be a triangle, having the side AB greater than the side AC; then will the angle ACB, opposite the greater side AB, be greater than the angle B, opposite the less side AC.

For, on the greater side AB, take the part AD equal to the less side AC, and join CD. Then, since BCD is a triangle, the outward angle ADC is greater than the inward opposite angle B (th. 8). But the angle ACD is equal to the said outward angle ADC, because AD is equal to AC (th. 3). Consequently the angle ACD also is greater than the angle B. And since the angle ACD is only a part of ACB, much more must the whole angle ACB be greater than the angle B. Q. E. D.

Again, conversely, if the angle C be greater than the angle B, then will the side AB, opposite the former, be greater than the side AC, opposite the latter.

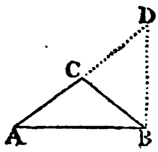
For, if AB be not greater than AC, it must be either *equal to it*, or less than it. But it cannot be equal, for

then

then the angle C would be equal to the angle B (th. 3), which it is not, by the supposition. Neither can it be less, for then the angle C would be less than the angle B, by the former part of this; which is also contrary to the supposition. The side AB, then, being neither equal to AC, nor less than it, must necessarily be greater. Q. E. D.

THEOREM X.

THE Sum of any Two Sides of a Triangle is Greater than the Third Side.



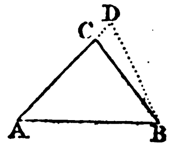
Let ABC be a triangle; then will the sum of any two of its sides be greater than the third side, as for instance, AC + CB greater than AB.

For, produce AC till CD be equal to CB, or AD equal to the sum of the two AC + CB; and join BD:—Then, because CD is equal to CB (by constr.), the angle D is equal to the angle CBD (th. 3). But the angle ABD is greater than the angle CBD, consequently it must also be greater than the angle D. And, since the greater side of any triangle is opposite to the greater angle (th. 9), the side AD (of the triangle ABD) is greater than the side AB. But AD is equal to AC and CD, or AC and CB, taken together (by constr.); therefore AC + CB is also greater than AB. Q. E. D.

Corol. The shortest distance between two points, is a single right line drawn from the one point to the other.

THEOREM XI.

THE Difference of any Two Sides of a Triangle, is Less than the Third Side.



Let ABC be a triangle; then will the difference of any two sides, as AB − AC, be less than the third side BC.

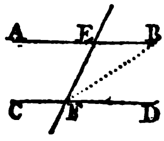
For, produce the less side AC to D, till AD be equal to the greater side AB, so that CD may be the difference of the two sides AB − AC; and join BD. Then, because AD is equal to AB (by constr.), the opposite angles D and ABD are equal (th. 3). But the angle CBD is less than the angle ABD, an

the equal angle D. And s

is opposite to the greater angle (th. 9), the side CD (of the triangle BCD) is less than the side BC. Q. E. D.

THEOREM XII.

WHEN a Line Intersects two Parallel Lines, it makes the Alternate Angles Equal to each other.



Let the line EF cut the two parallel lines AB, CD; then will the angle AEF be equal to the alternate angle EFD.

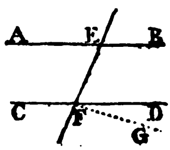
For if they are not equal, one of them must be greater than the other; let it be EFD for instance which is the greater, if possible; and conceive the line FB to be drawn; cutting off the part or angle EFB equal to the angle AEF; and meeting the line AB in the point B.

Then, since the outward angle AEF, of the triangle BEF, is greater than the inward opposite angle EFB (th. 8); and since these two angles also are equal (by the constr.) it follows, that those angles are both equal and unequal at the same time: which is impossible. Therefore the angle EFD is not unequal to the alternate angle AEF, that is, they are equal to each other. Q. E. D.

Corol. Right lines which are perpendicular to one, of two parallel lines, are also perpendicular to the other.

THEOREM XIII.

WHEN a Line, Cutting Two other Lines, makes the Alternate Angles Equal to each other, those two Lines are Parallel.



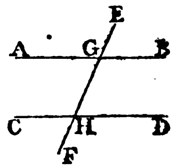
Let the line EF, cutting the two lines AB, CD, make the alternate angles AEF, DFE, equal to each other; then will AB be parallel to CD.

For if they be not parallel, let some other line, as FG, be parallel to AB. Then, because of these parallels, the angle AEF is equal to the alternate angle EFG (th. 12). But the angle AEF is equal to the angle EFD (by hyp.) Therefore the angle EFD is equal to the angle EFG (ax. 1); that is, a part is equal to the whole, which is impossible. Therefore no line but CD can be parallel to AB. Q. E. D.

Corol. Those lines which are perpendicular to the same line, are parallel to each other.

THEOREM XIV.

When a Line cuts two Parallel Lines, the Outward Angle is Equal to the Inward Opposite one, on the Same Side; and the two Inward Angles, on the Same Side, equal to two Right Angles.



Let the line EF cut the two parallel lines AB, CD; then will the outward angle EGB be equal to the inward opposite angle GHD, on the same side of the line EF; and the two inward angles BGH, GHD, taken together, will be equal to two right angles.

For, since the two lines AB, CD, are parallel, the angle AGH is equal to the alternate angle GHD, (th. 12). But the angle AGH is equal to the opposite angle EGB (th. 7). Therefore the angle EGB is also equal to the angle GHD (ax. 1). Q. E. D.

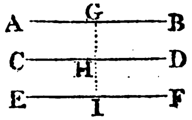
Again, because the two adjacent angles EGB, BGH, are together equal to two right angles (th. 6); of which the angle EGB has been shown to be equal to the angle GHD; therefore the two angles BGH, GHD, taken together, are also equal to two right angles.

Corol. 1. And, conversely, if one line meeting two other lines, make the angles on the same side of it equal, those two lines are parallels.

Corol. 2. If a line, cutting two other lines, make the sum of the two inward angles, on the same side, less than two right angles, those two lines will not be parallel, but will meet each other when produced.

THEOREM XV.

Those Lines which are Parallel to the Same Line, are Parallel to each other.



Let the Lines AB, CD, be each of them parallel to the line EF; then shall the lines AB, CD, be parallel to each other.

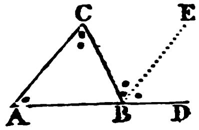
For, let the line GI be perpendicular to EF. Then will this line be also perpendicular to both the lines AB, CD (corol. th. 12), and consequently the two lines AB, CD, are parallels (corol. th. 13).

Q. E. D.

THE-

THEOREM XVI.

When one Side of a Triangle is produced, the Outward Angle is equal to both the Inward Opposite Angles taken together.

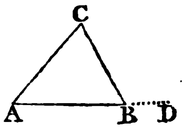


Let the side AB, of the triangle ABC, be produced to D; then will the outward angle CBD be equal to the sum of the two inward opposite angles A and C.

For, conceive BE to be drawn parallel to the side AC of the triangle. Then BC, meeting the two parallels AC, BE, makes the alternate angles C and CBE equal (th. 12). And AD, cutting the same two parallels AC, BE, makes the inward and outward angles on the same side, A and EBD, equal to each other (th. 14). Therefore, by equal additions, the sum of the two angles A and C, is equal to the sum of the two CBE and EBD, that is, to the whole angle CBD (by ax. 2). Q. E. D.

THEOREM XVII.

In any Triangle, the sum of all the Three Angles is equal to Two Right Angles.



Let ABC be any plane triangle; then the sum of the three angles A + B + C is equal to two right angles.

For, let the side AB be produced to D. Then the outward angle CBD is equal to the sum of the two inward opposite angles A + C (th. 16). To each of these equals add the inward angle B, then will the sum of the three inward angles A + B + C be equal to the sum of the two adjacent angles ABC + CBD (ax. 2). But the sum of these two last adjacent angles is equal to two right angles (th. 6). Therefore also the sum of the three angles of the triangle A + B + C is equal to two right angles (ax. 1). Q. E. D.

Corol. 1. If two angles in one triangle, be equal to two angles in another triangle, the third angles will also be equal (ax. 3), and the two triangles equiangular.

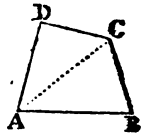
Corol. 2. If one angle in one triangle, be e[illegible] angle in another, the sums of the remaining an[illegible] be equal (ax. 3).

Corol. 3. If one angle of a triangle be right, the sum of the other two will also be equal to a right angle, and each of them singly will be acute, or less than a right angle.

Corol. 4. The two least angles of every triangle are acute, or each less than a right angle.

THEOREM XVIII.

In any Quadrangle, the sum of all the Four Inward Angles, is equal to Four Right Angles.



Let ABCD be a quadrangle; then the sum of the four inward angles, A + B + C + D is equal to four right angles.

Let the diagonal AC be drawn, dividing the quadrangle into two triangle, ABC, ADC. Then, because the sum of the three angles of each of these triangles is equal to two right angles (th. 17); it follows, that the sum of all the angles of both triangles, which make up the four angles of the quadrangle, must be equal to four right angles (ax. 2).

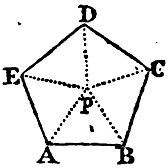
Q. E. D.

Corol. 1. Hence, if three of the angles be right ones, the fourth will also be a right angle.

Corol. 2. And, if the sum of two of the four angles be equal to two right angles, the sum of the remaining two will also be equal to two right angles.

THEOREM XIX.

In any figure whatever, the Sum of all the Inward Angles, taken together, is equal to Twice as many Right Angles, wanting four, as the Figure has Sides.



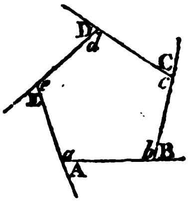
Let ABCDE be any figure; then the sum of all its inward angles, A + B + C + D + E, is equal to twice as many right angles, wanting four, as the figure has sides.

For, from any point P, within it, draw lines PA, PB, PC, &c, to all the angles, dividing the polygon into as many triangles as it has sides. Now the sum of the three angles of each c[illegible] ...ual to two right angles (th. 17); ...s of all the triangles is equal ... the figure has sides. But ...oint P, which are so many

many of the angles of the triangles, but no part of the inward angles of the polygon, is equal to four right angles (corol. 3, th. 6), and must be deducted out of the former sum. Hence it follows that the sum of all the inward angles of the polygon alone, A + B + C + D + E, is equal to twice as many right angles as the figure has sides, wanting the said four right angles. Q. E. D.

THEOREM XX.

When every Side of any Figure is produced out, the Sum of all the Outward Angles thereby made, is equal to Four Right Angles.

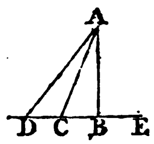


Let A, B, C, &c, be the outward angles of any polygon, made by producing all the sides; then will the sum A + B + C + D + E, of all those outward angles, be equal to four right angles.

For every one of these outward angles, together with its adjacent inward angle, make up two right angles, as A + a equal to two right angles, being the two angles made by one line meeting another (th. 6). And there being as many outward, or inward angles, as the figure has sides; therefore the sum of all the inward and outward angles, is equal to twice as many right angles as the figure has sides. But the sum of all the inward angles, with four right angles, is equal to twice as many right angles as the figure has sides (th. 19). Therefore the sum of all the inward and all the outward angles, is equal to the sum of all the inward angles and four right angles (by ax. 1). From each of these take away all the inward angles, and there remains all the outward angles equal to four right angles (by ax. 3).

THEOREM XXI.

A Perpendicular is the Shortest Line that can be drawn from a Given Point to an Indefinite Line. And, of any other Lines drawn from the same Point, those that are Nearest the Perpendicular, are Less than those More Remote.



If AB, AC, AD, &c, be lines drawn from the given point A, to the indefinite line DE, of which AB is perpendicular. Then shall the perpendicular AB be less than AC, and AC less than AD, &c.

For, the angle B being a right one, the

angle

angle C is acute (by cor. 3, th. 17), and therefore less than the angle B. But the less angle of a triangle is subtended by the less side (th. 9). Therefore the side AB is less than the side AC.

Again, the angle ACB being acute, as before, the adjacent angle ACD will be obtuse (by th. 6); consequently the angle D is acute (corol. 3, th. 17), and therefore is less than the angle C. And since the less side is opposite to the less angle, therefore the side AC is less than the side AD.

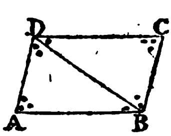
Q. E. D.

Corol. A perpendicular is the least distance of a given point from a line.

THEOREM XXII.

THE Opposite Sides and Angles of any Parallelogram are equal to each other; and the Diagonal divides it into two Equal Triangles.

Let ABCD be a parallelogram, of which the diagonal is BD; then will its opposite sides and angles be equal to each other, and the diagonal BD will divide it into two equal parts, or triangles.



For, since the sides AB and DC are parallel, as also the sides AD and BC (defin. 32), and the line BD meets them; therefore the alternate angles are equal (th. 12), namely, the angle ABD to the angle CDB, and the angle ADB to the angle CBD. Hence the two triangles, having two angles in the one equal to two angles in the other, have also their third angles equal (cor. 1, th. 17), namely, the angle A equal to the angle C, which are two of the opposite angles of the parallelogram.

Also, if to the equal angles ABD, CDB, be added the equal angles CBD, ADB, the wholes will be equal (ax. 2), namely, the whole angle ABC to the whole ADC, which are the other two opposite angles of the parallelogram.

Q. E. D.

Again, since the two triangles are mutually equiangular, and have a side in each equal, viz. the common side BD; therefore the two triangles are identical (th. 2), or equal in all respects, namely, the side AB equal to the opposite side DC, and AD equal to the opposite side BC, and the whole triangle ABD equal to the whole triangle BCD. Q. E. D.

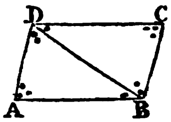
Corol.

Corol. 1. Hence, if one angle of a parallelogram be a right angle, all the other three will also be right angles, and the parallelogram a rectangle.

Corol. 2. Hence also, the sum of any two adjacent angles of a parallelogram is equal to two right angles.

THEOREM XXIII.

EVERY Quadrilateral, whose Opposite Sides are Equal, is a Parallelogram, or has its Opposite Sides Parallel.



Let ABCD be a quadrangle, having the opposite sides equal, namely, the side AB equal to DC, and AD equal to BC; then shall these equal sides be also parallel, and the figure a parallelogram.

For, let the diagonal BD be drawn. Then, the triangles, ABD, CBD, being mutually equilateral (by hyp.), they are also mutually equiangular (th. 5), or have their corresponding angles equal; consequently the opposite sides are parallel (th. 13); viz. the side AB parallel to DC, and AD parallel to BC, and the figure is a parallelogram. Q. E. D.

THEOREM XXIV.

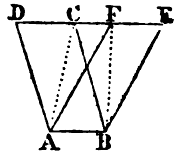
THOSE Lines which join the Corresponding Extremes of two Equal and Parallel Lines, are themselves Equal and Parallel.

Let AB, DC, be two equal and parallel lines; then will the lines AD, BC, which join their extremes, be also equal and parallel. [See the fig. above.]

For, draw the diagonal BD. Then, because AB and DC are parallel (by hyp.), the angle ABD is equal to the alternate angle BDC (th. 12). Hence then, the two triangles having two sides and the contained angles equal, viz. the side AB equal to the side DC, and the side BD common, and the contained angle ABD equal to the contained angle BDC, they have the remaining sides and angles also respectively equal (th. 1); consequently AD is equal to BC, and also parallel to it (th. 12). Q. E. D.

THEOREM XXV.

PARALLELOGRAMS, as also Triangles, standing on the Same Base, and between the Same Parallels, are equal to each other.



Let ABCD, ABEF, be two parallelograms, and ABC, ABF, two triangles, standing on the same base AB, and between the same parallels AB, DE; then will the parallelogram ABCD be equal to the parallelogram ABEF, and the triangle ABC equal to the triangle ABF.

For, since the line DE cuts the two parallels AF, BE, and the two AD, BC, it makes the angle E equal to the angle AFD, and the angle D equal to the angle BCE (th. 14); the two triangles ADF, BCE, are therefore equiangular (cor. 1, th. 17); and having the two corresponding sides, AD, BC, equal (th. 22), being opposite sides of a parallelogram, these two triangles are identical, or equal in all respects (th. 2). If each of these equal triangles then be taken from the whole space ABED, there will remain the parallelogram ABEF in the one case, equal to the parallelogram ABCD in the other (by ax. 3).

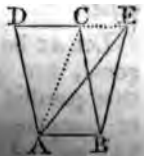
Also the triangles ABC, ABF, on the same base AB, and between the same parallels, are equal, being the halves of the said equal parallelograms (th. 22). Q. E. D.

Corol. 1. Parallelograms, or triangles, having the same base and altitude, are equal. For the altitude is the same as the perpendicular or distance between the two parallels, which is every where equal, by the definition of parallels.

Corol. 2. Parallelograms, or triangles, having equal bases and altitudes, are equal. For, if the one figure be applied with its base on the other, the bases will coincide or be the same, because they are equal: and so the two figures, having the same base and altitude, are equal.

THEOREM XXVI.

If a Parallelogram and a Triangle stand on the Same Base, and between the Same Parallels, the Parallelogram will be Double the Triangle, or the Triangle Half the Parallelogram.



Let ABCD be a parallelogram, and ABE a triangle, on the same base AB, and between the same parallels AB, DE; then will the parallelogram ABCD be double the triangle ABE, or the triangle half the parallelogram.

For, draw the diagonal AC of the parallelogram, dividing it into two equal parts (th. 22). Then because the triangles ABC,

ABE,

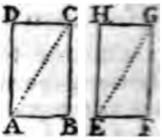
ABE, on the same base, and between the same parallels, are equal (th. 25); and because the one triangle ABC is half the parallelogram ABCD (th. 22), the other equal triangle ABE is also equal to half the same parallelogram ABCD. Q. E. D.

Corol. 1. A triangle is equal to half a parallelogram of the same base and altitude, because the altitude is the perpendicular distance between the parallels, which is every where equal, by the definition of parallels.

Corol. 2. If the base of a parallelogram be half that of a triangle, of the same altitude, or the base of the triangle be double that of the parallelogram, the two figures will be equal to each other.

THEOREM XXVII.

RECTANGLES that are contained by Equal Lines, are Equal to each other.



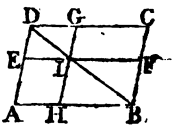
Let BD, FH, be two rectangles, having the sides AB, BC, equal to the sides EF, FG, each to each; then will the rectangle BD be equal to the rectangle FH.

For, draw the two diagonals AC, EG, dividing the two parallelograms each into two equal parts. Then the two triangles ABC, EFG, are equal to each other (th. 1), because they have the two sides AB, BC, and the contained angle B, equal to the two sides EF, FG, and the contained angle F (by hyp). But these equal triangles are the halves of the respective rectangles. And because the halves, or the triangles, are equal, the wholes, or the rectangles DB, HF, are also equal (by ax. 6). Q. E. D.

Corol. The squares on equal lines are also equal; for every square is a species of rectangle.

THEOREM XXVIII.

THE Complements of the Parallelograms, which are about the Diagonal of any Parallelogram, are equal to each other.



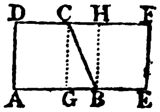
Let AC be a parallelogram, BD a diagonal, EIF parallel to AB or DC, and GIH parallel to AD or BC, making AI, IC complements to the parallelograms EG, HF, which are about the diagonal DB: then will the complement AI be equal to the complement IC.

For,

For, since the diagonal DB bisects the three parallelograms AC, EG, HF (th. 22); therefore, the whole triangle DAB being equal to the whole triangle DCB, and the parts DEI, IHB, respectively equal to the parts DGI, IFB, the remaining parts AI, IC, must also be equal (by ax. 3). Q. E. D.

THEOREM XXIX.

A Trapezoid, or Trapezium having two Sides Parallel, is equal to Half a Parallelogram, whose Base is the Sum of those two Sides, and its Altitude the Perpendicular Distance between them.

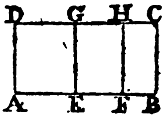


Let ABCD be the trapezoid, having its two sides AB, DC, parallel; and in AB produced take BE equal to DC, so that AE may be the sum of the two parallel sides; produce DC also, and let EF, GC, BH, be all three parallel to AD. Then is AF a parallelogram of the same altitude with the trapezoid ABCD, having its base AE equal to the sum of the parallel sides of the trapezoid; and it is to be proved that the trapezoid ABCD is equal to half the parallelogram AF.

Now, since triangles, or parallelograms, of equal bases and altitude, are equal (corol. 2, th. 25), the parallelogram DG is equal to the parallelogram HE, and the triangle CGB equal to the triangle CHB; consequently the line BC bisects, or equally divides, the parallelogram AF, and ABCD is the half of it. Q. E. D.

THEOREM XXX.

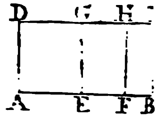
The Sum of all the Rectangles contained under one Whole Line, and the several Parts of another Line, any way divided, is Equal to the Rectangle contained under the Two Whole Lines.



Let AD be the one line, and AB the other, divided into the parts AE, EF, FB; then will the rectangle contained by AD and AB, be equal to the sum of the rectangles of AD and AE, and AD and EF, and AD and FB: thus expressed, AD . AB = AD . AE + AD . EF + AD . FB.

For, make the rectangle AC of the two whole lines AD, AB; and draw EG, FH, perpendicular to AB, or parallel to AD, to which they are equal (th. 22). Then the whole rectangle AC is made up of all the other rectangles

ABE, on the same ... equal (th. 25); and b... B; parallelogram ABCD ... AD also equal to half the ... because



..., EG, FH.

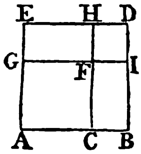
Corol. 1. A trian... same base and al... cular distance be... equal, by the d... ...B is equal to the sum of all ... AD . EF, AD . FB. Q. E. D.

Corol. 2. If ... triangle, of ... double ... equal t... ... divided into any two parts; the ... is equal to both the rectangles of ... of the parts.

THEOREM XXXI.

... Sum of two Lines, is greater than the ...ares, by Twice the Rectangle of the said RECTA... ...e Square of a whole Line, is equal to the ... two Parts, together with Twice the Rectangle

Let ... the ... line AB be the sum of any two ..., ... CB: then will the square of AB ... to the squares of AC, CB, together ... the rectangle of AC . CB. That ... $AC^2 + CB^2 + 2AC . CB$.



... Let ABDE be the square on the sum ... line AB, and ACFG the square ... part AC. Produce CF and GF to the other sides at H ...

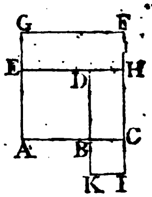
... the lines CH, GI, which are equal, being each ... to the sides of the square AB or BD (th. 22), take the ... CF, GF, which are also equal, being the sides of the ... AF, and there remains FH equal to FI, which are ... equal to DH, DI, being the opposite sides of a parallelo... Hence the figure HI is equilateral: and it has all ... angles right ones (corol. 1, th. 22); it is therefore a ... are on the line FI, or the square of its equal CB. Also ... figures EF, FB, are equal to two rectangles under AC ... CB, because GF is equal to AC, and FH or FI equal ... CB. But the whole square AD is made up of the four ... ares, viz. the two squares AF, FD, and the two equal rect... ...gles EF, FB. That is, the square of AB is equal to the ... ares of AC, CB, together with twice the rectangle of AC, ... Q. E. D.

...ol. Hence, if a line be divided into two equal parts; ... square of the whole line, will be equal to four times the ... are of half the line.

THEOREM

THEOREM XXXII.

The Square of the Difference of two lines, is less than the Sum of their Squares, by Twice the Rectangle of the said Lines.



Let AC, BC, be any two lines, and AB their difference: then will the square of AB be less than the squares of AC, BC, by twice the rectangle of AC and BC. Or, $AB^2 = AC^2 + BC^2 - 2AC . BC$.

For, let ABDE be the square on the difference AB, and ACFG the square on the line AC. Produce ED to H; also produce DB and HC, and draw KI, making BI the square of the other line BC.

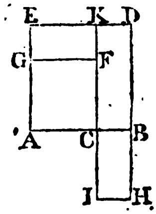
Now it is visible that the square AD is less than the two squares AF, BI, by the two rectangles EF, DI. But GF is equal to the one line AC, and GE or FH is equal to the other line BC; consequently the rectangle EF, contained under EG and GF, is equal to the rectangle of AC and BC.

Again, FH being equal to CI or BC or DH, by adding the common part HC, the whole HI will be equal to the whole FC, or equal to AC; and consequently the figure DI is equal to the rectangle contained by AC and BC.

Hence the two figures EF, DI, are two rectangles of the two lines AC, BC; and consequently the square of AB is less than the squares of AC, BC, by twice the rectangle AC . BC. Q. E. D.

THEOREM XXXIII.

The Rectangle under the Sum and Difference of two Lines, is equal to the Difference of the Squares of those Lines.



Let AB, AC, be any two unequal lines; then will the difference of the squares of AB, AC, be equal to a rectangle under their sum and difference. That is,

$AB^2 - AC^2 = \overline{AB + AC} . \overline{AB - AC}$.

For, let ABDE be the square of AB, and ACFG the square of AC. Produce DB till BH be equal to AC; draw HI parallel to AB or ED, and produce FC both ways to I and K.

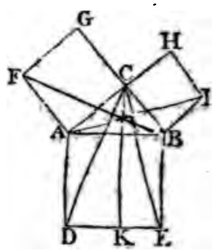
Then the difference of the two squares AD, AF, is evi-

 dently

dently the two rectangles EF, KB. But the rectangles EF, BI, are equal, being contained under equal lines; for EK and BH are each equal to AC, and GE is equal to CB, being each equal to the difference between AB and AC, or their equals AE and AG. Therefore the two EF, KB, are equal to the two KB, BI, or to the whole KH; and consequently KH is equal to the difference of the squares AD, AF. But KH is a rectangle contained by DH, or the sum of AB and AC, and by KD, or the difference of AB and AC. Therefore the difference of the squares of AB, AC, is equal to the rectangle under their sum and difference. Q. E. D.

THEOREM XXXIV.

In any Right-angled Triangle, the Square of the Hypothenuse, is equal to the Sum of the Squares of the other two Sides.



Let ABC be a right-angled triangle, having the right angle C; then will the square of the hypothenuse AB, be equal to the sum of the squares of the other two sides AC, CB. Or $AB^2 = AC^2 + BC^2$.

For, on AB describe the square AE, and on AC, CB, the squares AG, BH; then draw CK parallel to AD or BE; and join AI, BF, CD, CE.

Now, because the line AC meets the two CG, CB, so as to make two right angles, these two form one straight line GB (corol. 1, th. 6). And because the angle FAC is equal to the angle DAB, being each a right angle, or the angle of a square; to each of these equals add the common angle BAC, so will the whole angle or sum FAB, be equal to the whole angle or sum CAD. But the line FA is equal to the line AC, and the line AB to the line AD, being sides of the same square; so that the two sides FA, AB, and their included angle FAB, are equal to the two sides CA, AD, and the contained angle CAD, each to each; therefore the whole triangle AFB is equal to the whole triangle ACD (th. 1).

But the square AG is double the triangle AFB, on the same base FA, and between the same parallels FA, GB (th. 26); in like manner, the parallelogram AK is double the triangle ACD, on the same base AD, and between the same parallels AD, CK. And since the doubles of equal things, are equal (by ax. 6); therefore the square AG is equal to the parallelogram AK.

In

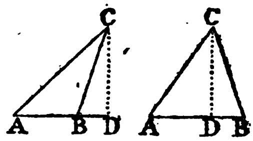
In like manner, the other square BH is proved equal to the other parallelogram BK. Consequently the two squares AG and BH together, are equal to the two parallelograms AK and BK together, or to the whole square AE. That is, the sum of the two squares on the two less sides, is equal to the square on the greatest side. Q. E. D.

Corol. 1. Hence, the square of either of the two less sides, is equal to the difference of the squares of the hypothenuse and the other side (ax. 3); or, equal to the rectangle contained by the sum and difference of the said hypothenuse and other side (th. 33).

Corol. 2. Hence also, if two right-angled triangles have two sides of the one equal to two corresponding sides of the other; their third sides will also be equal, and the triangles identical.

THEOREM XXXV.

In any Triangle, the Difference of the Squares of the two Sides, is Equal to the Difference of the Squares of the Segments of the Base, or of the two Lines, or Distances, included between the Extremes of the Base and the Perpendicular.



Let ABC be any triangle, having CD perpendicular to AB; then will the difference of the squares of AC, BC, be equal to the difference of the squares of AD, BD; that is, $AC^2 - BC^2 = AD^2 - BD^2$.

For, since AC^2 is equal to $AD^2 + CD^2$ and BC^2 is equal to $BD^2 + CD^2$ } (by th. 34);
Theref. the difference between AC^2 and BC^2,
is equal to the difference between $AD^2 + CD^2$ and $BD^2 + CD^2$,
or equal to the difference between AD^2 and BD^2,
by taking away the common square CD^2 Q. E. D.

Corol. The rectangle of the sum and difference of the two sides of any triangle, is equal to the rectangle of the sum and difference of the distances between the perpendicular and the two extremes of the base, or equal to the rectangle of the base and the difference or sum of the segments, according as the perpendicular falls within or without the triangle.

That

That is, $\overline{AC + BC} \cdot \overline{AC - BC} = \overline{AD + BD} \cdot \overline{AD - BD}$

Or, $\overline{AC + BC} \cdot \overline{AC - BC} = AB \cdot \overline{AD - BD}$ in the 2d figure.

And $\overline{AC + BC} \cdot \overline{AC - BC} = AB \cdot \overline{AD + BD}$ in the 1st figure.

THEOREM XXXVI.

In any Obtuse-angled Triangle, the Square of the Side subtending the Obtuse Angle, is Greater than the Sum of the Squares of the other two Sides, by Twice the Rectangle of the Base and the Distance of the Perpendicular from the Obtuse Angle.

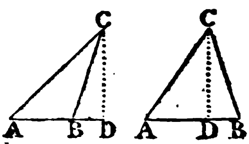
Let ABC be a triangle, obtuse angled at B, and CD perpendicular to AB; then will the square of AC be greater than the squares of AB, BC, by twice the rectangle of AB, BD. That is, $AC^2 = AB^2 + BC^2 + 2AB \cdot BD$. See the 1st fig. above, or below.

For, since the square of the whole line AD is equal to the squares of the parts AB, BD, with twice the rectangle of the same parts AB, BD (th. 31); if to each of these equals there be added the square of CD, then the squares of AD, CD, will be equal to the squares of AB, BD, CD, with twice the rectangle of AB, BD (by ax. 2).

But the squares of AD, CD, are equal to the square of AC; and the squares of BD, CD, equal to the square of BC (th. 34); therefore the square of AC is equal to the squares of AB, BC, together with twice the rectangle of AB, BD. Q. E. D.

THEOREM XXXVII.

In any Triangle, the Square of the Side subtending an Acute Angle, is Less than the Squares of the Base and the other Side, by Twice the Rectangle of the Base and the Distance of the Perpendicular from the Acute Angle.



Let ABC be a triangle, having the angle A acute, and CD perpendicular to AB; then will the square of BC, be less than the squares of AB, AC, by twice the rectangle of AB, AD. That is, $BC^2 = AB^2 + AC^2 - 2AB \cdot AD$.

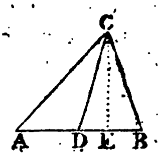
For, in fig. 1, AC^2 is $= BC^2 + AB^2 + 2AB \cdot BD$ (th. 36).
To each of these equals add the square of AB,
then is $AB^2 + AC^2 = BC^2 + 2AB^2 + 2AB \cdot BD$ (ax. 2),
or $= BC^2 + 2AB \cdot AD$ (th. 30).

Q. E. D.

Again, in fig. 2, AC^2 is $= AD^2 + DC^2$ (th. 34),
And $AB^2 = AD^2 + DB^2 + 2AD \cdot DB$ (th. 31).
Theref. $AB^2 + AC^2 = BD^2 + DC^2 + 2AD^2 + 2AD \cdot DB$ (ax. 2),
or $= BC^2 + 2AD^2 + 2AD \cdot DB$ (th. 34),
or $= BC^2 + 2AB \cdot AD$ (th. 30). Q. E. D.

THEOREM XXXVIII.

In any Triangle, the Double of the Square of a Line drawn from the Vertex to the Middle of the Base, together with Double the Square of the Half Base, is Equal to the Sum of the Squares of the other Two Sides.

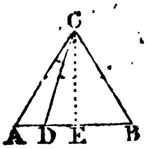


Let ABC be a triangle, and CD the line drawn from the vertex to the middle of the base AB, bisecting it into the two equal parts AD, DB; then will the sum of the squares of AC, CB, be equal to twice the sum of the squares of CD, BD; or $AC^2 + CB^2 = 2CD^2 + 2DB^2$.

For, let CE be perpendicular to the base AB. Then, since (by th. 36) AC^2 exceeds the sum of the two squares AD^2 and CD^2 (or BD^2 and CD^2) by the double rectangle $2AD \cdot DE$ (or $2BD \cdot DE$); and since (by th. 37) BC^2 is less than the same sum by the said double rectangle; it is manifest that both AC^2 and BC^2 together, must be equal to that sum twice taken; the excess on the one part making up the defect on the other. Q. E. D.

THEOREM XXXIX.

In an Isosceles Triangle, the Square of a Line drawn from the Vertex to any Point in the Base, together with the Rectangle of the Segments of the Base, is equal to the Square of one of the Equal Sides of the Triangle.

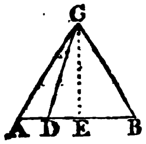


Let ABC be the isosceles triangle, and CD a line drawn from the vertex to any point D in the base: then will the square of AC, be equal to the square of CD, together with the rectangle of AD and DB. That is, $AC^2 = CD^2 + AD \cdot DB$.

For,

For, let CE bisect the vertical angle; then will it also bisect the base AB perpendicularly, making $AE = EB$ (cor. 1, th. 3).

But, in the triangle ACD, obtuse angled at D, the square AC^2 is $= CD^2 + AD^2 + 2AD \cdot DE$ (th. 36),

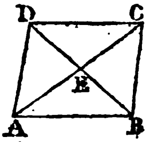


$$\text{or } = CD^2 + AD \cdot \overline{AD + 2DE} \text{ (th. 30)},$$
$$\text{or } = CD^2 + AD \cdot \overline{AE + DE},$$
$$\text{or } = CD^2 + AD \cdot \overline{BE + DE},$$
$$\text{or } = CD^2 + AD \cdot DB.$$

Q. E. D.

THEOREM XL.

In any Parallelogram, the two Diagonals Bisect each other; and the Sum of their Squares is equal to the Sum of the Squares of all the Four Sides of the Parallelogram.



Let ABCD be a parallelogram, whose diagonals intersect each other in E: then will AE be equal to EC, and BE to ED; and the sum of the squares of AC, BD, will be equal to the sum of the squares of AB, BC, CD, DA. That is,

$$AE = EC, \text{ and } BE = ED,$$
$$\text{and } AC^2 + BD^2 = AB^2 + BC^2 + CD^2 + DA^2.$$

For, the triangles AEB, DEC, are equiangular, because they have the opposite angles at E equal (th. 7), and the two lines AC, BD, meeting the parallels AB, DC, make the angle BAE equal to the angle DCE, and the angle ABE equal to the angle CDE, and the side AB equal to the side DC (th. 22); therefore these two triangles are identical, and have their corresponding sides equal (th. 2), viz. $AE = EC$, and $BE = ED$.

Again, since AC is bisected in E, the sum of the squares $AD^2 + DC^2 = 2AE^2 + 2DE^2$ (th. 33).

In like manner, $AB^2 + BC^2 = 2AE^2 + 2BE^2$ or $2DE^2$.

Theref. $AB^2 + BC^2 + CD^2 + DA^2 = 4AE^2 + 4DE^2$ (ax. 2).

But, because the square of a whole line is equal to 4 times the square of half the line (cor. th. 31), that is, $AC^2 = 4AE^2$, and $BD^2 = 4DE^2$.

Theref. $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$ (ax. 1).

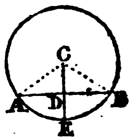
Q. E. D.

THEOREM

THEOREM XLI.

If a Line, drawn through or from the Centre of a Circle, Bisect a Chord, it will be Perpendicular to it; or, if it be Perpendicular to the Chord, it will Bisect both the Chord and the Arc of the Chord.

Let AB be any chord in a circle, and CD a line drawn from the centre C to the chord. Then, if the chord be bisected in the point D, CD will be perpendicular to AB.



For, draw the two radii CA, CB. Then, the two triangles ACD, BCD, having CA equal to CB (def. 44), and CD common, also AD equal to DB (by hyp.); they have all the three sides of the one, equal to all the three sides of the other, and so have their angles also equal (th. 5). Hence then, the angle ADC being equal to the angle BDC, these angles are right angles, and the line CD is perpendicular to AB (def. 11).

Again, if CD be perpendicular to AB, then will the chord AB be bisected at the point D, or have AD equal to DB; and the arc AEB bisected in the point E, or have AE equal EB.

For, having drawn CA, CB, as before. Then, in the triangle ABC, because the side CA is equal to the side CB, their opposite angles A and B are also equal (th. 3). Hence then, in the two triangles ACD, BCD, the angle A is equal to the angle B, and the angles at D are equal (def. 11); therefore their third angles are also equal (corol. 1, th. 17). And having the side CD common, they have also the side AD equal to the side DB (th. 2).

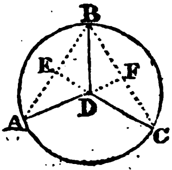
Also, since the angle ACE is equal to the angle BCE, the arc AE, which measures the former (def. 57), is equal to the arc BE, which measures the latter, since equal angles must have equal measures.

Corol. Hence a line bisecting any chord at right angles, passes through the centre of the circle.

THEOREM XLII.

If More than Two Equal Lines can be drawn from any Point within a Circle to the Circumference, that Point will be the Centre.

Let



Let ABC be a circle, and D a point within it: then if any three lines, DA, DB, DC, drawn from the point D to the circumference, be equal to each other, the point D will be the centre.

For, draw the chords AB, BC, which let be bisected in the points E, F, and join DE, DF.

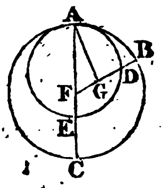
Then, the two triangles, DAE, DBE, have the side DA equal to the side DB by supposition, and the side AE equal to the side EB by hypothesis, also the side DE common: therefore these two triangles are identical, and have the angles at E equal to each other (th. 5); consequently DE is perpendicular to the middle of the chord AB (def. 11), and therefore passes through the centre of the circle (corol. th. 41).

In like manner, it may be shown that DF passes through the centre. Consequently the point D is the centre of the circle, and the three equal lines DA, DB, DC, are radii.

Q. E. D.

THEOREM XLIII.

If two Circles touch one another Internally, the Centres of the Circles and the Point of Contact will be all in the Same Right Line.



Let the two circles ABC, ADE, touch one another internally in the point A; then will the point A and the centres of those circles be all in the same right line.

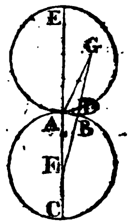
For, let F be the centre of the circle ABC, through which draw the diameter AFC. Then, if the centre of the other circle can be out of this line AC, let it be supposed in some other point as G; through which draw the line FG cutting the two circles in B and D.

Now, in the triangle AFG, the sum of the two sides FG, GA, is greater than the third side AF (th. 10), or greater than its equal radius FB. From each of these take away the common part FG, and the remainder GA will be greater than the remainder GB. But the point G being supposed the centre of the inner circle, its two radii, GA, GD, are equal to each other; consequently GD will also be greater than GB. But ADE being the inner circle, GD is necessarily less

less than GB. So that GD is both greater and less than GB; which is absurd. Consequently the centre G cannot be out of the line AFC. Q. E. D.

THEOREM XLIV.

If two Circles Touch one another Externally, the Centres of the Circles and the Point of Contact will be all in the Same Right Line.



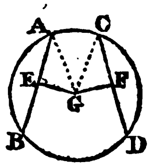
LET the two circles ABC, ADE, touch one another externally at the point A; then will the point of contact A and the centres of the two circles be all in the same right line.

For, let F be the centre of the circle ABC, through which draw the diameter AFC, and produce it to the other circle at E. Then, if the centre of the other circle ADE can be out of the line FE, let it, if possible, be supposed in some other point as G; and draw the lines AG, FBDG, cutting the two circles in B and D.

Then, in the triangle AFG, the sum of the two sides AF, AG, is greater than the third side FG (th. 10). But, F and G being the centres of the two circles, the two radii GA, GD, are equal, as are also the two radii AF, FB. Hence the sum of GA, AF, is equal to the sum of GD, BF; and therefore this latter sum also, GD, BF, is greater than GF, which is absurd. Consequently the centre G cannot be out of the line EF. Q. E. D.

THEOREM XLV.

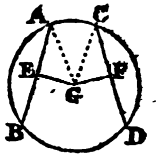
Any Chords in a Circle, which are Equally Distant from the Centre, are Equal to each other; or if they be Equal to each other, they will be Equally Distant from the Centre.



LET AB, CD, be any two chords at equal distances from the centre G; then will these two chords AB, CD, be equal to each other.

For, draw the two radii GA, GC, and the two perpendiculars GE, GF, which are the equal distances from the centre G. Then, the two right-angled triangles, GAE, GCF, having the side GA equal the side GC, and the side GE equal the side

side GF, and the angle at E equal to the angle at F, therefore the two triangles GAE, GCF, are identical (cor. 2, th. 34), and have the line AE equal the line CF. But AB is the double of AE, and CD is the double of CF (th. 41); therefore AB is equal to CD (by ax. 6). Q. E. D.



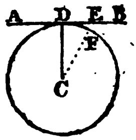
Again, if the chord AB be equal to the chord CD; then will their distances from the centre, GE, GF, also be equal to each other.

For, since AB is equal CD by supposition, the half AE is equal the half CF. Also the radii GA, GC, being equal, as well as the right angles E and F, therefore the third sides are equal (cor. 2, th. 34), or the distance GE equal the distance GF. Q. E. D.

THEOREM XLVI.

A Line Perpendicular to the Extremity of a Radius, is a Tangent to the Circle.

LET the line ADB be perpendicular to the radius CD of a circle; then shall AB touch the circle in the point D only.



For, from any other point E in the line AB draw CFE to the centre, cutting the circle in F.

Then, because the angle D, of the triangle CDE, is a right angle, the angle at E is acute (th. 17, cor. 3), and consequently less than the angle D. But the greater side is always opposite to the greater angle (th. 9); therefore the side CE is greater than the side CD, or greater than its equal CF. Hence the point E is without the circle; and the same for every other point in the line AB. Consequently the whole line is without the circle, and meets it in the point D only.

THEOREM

THEOREM XLVII.

When a Line is a Tangent to a Circle, a Radius drawn to the Point of Contact is Perpendicular to the Tangent.

LET the line AB touch the circumference of a circle at the point D; then will the radius CD be perpendicular to the tangent AB. [See the last figure.]

For, the line AB being wholly without the circumference except at the point D, every other line, as CE drawn from the centre C to the line AB, must pass out of the circle to arrive at this line. The line CD is therefore the shortest that can be drawn from the point C to the line AB, and consequently (th. 21) it is perpendicular to that line.

Corol. Hence, conversely, a line drawn perpendicular to a tangent, at the point of contact, passes through the centre of the circle.

THEOREM XLVIII.

The Angle formed by a Tangent and Chord is Measured by Half the Arc of that Chord.

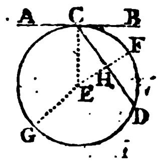
LET AB be a tangent to a circle, and CD a chord drawn from the point of contact C; then is the angle BCD measured by half the arc CFD, and the angle ACD measured by half the arc CGD.

For, draw the radius EC to the point of contact, and the radius EF perpendicular to the chord at H.

Then, the radius EF, being perpendicular to the chord CD, bisects the arc CFD (th. 41). Therefore CF is half the arc CFD.

In the triangle CEH, the angle H being a right one, the sum of the two remaining angles E and C is equal to a right angle (corol. 3, th. 17), which is equal to the angle BCE, because the radius CE is perpendicular to the tangent. From each of these equals take away the common part or angle C, and there remains the angle E equal to the angle BCD. But the angle E is measured by the arc CF (def. 57), which is the half of CFD; therefore the equal angle BCD must also have the same measure, namely, half the arc CFD of the chord CD.

Again,



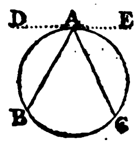
Again, the line GEF, being perpendicular to the chord CD, bisects the arc CGD (th. 41). Therefore CG is half the arc CGD. Now, since the line CE, meeting FG, makes the sum of the two angles at E equal to two right angles (th. 6), and the line CD makes with AB the sum of the two angles at C equal to two right angles; if from these two equal sums there be taken away the parts or angles CEH and BCH, which have been proved equal, there remains the angle CEG equal to the angle ACH. But the former of these, CEG, being an angle at the centre, is measured by the arc CG (def. 57); consequently the equal angle ACD must also have the same measure CG, which is half the arc CGD of the chord CD. Q. E. D.

Corol. 1. The sum of two right angles is measured by half the circumference. For the two angles BCD, ACD, which make up two right angles, are measured by the arcs CF, CG, which make up half the circumference, FG being a diameter.

Corol. 2. Hence also one right angle must have for its measure a quarter of the circumference, or 90 degrees.

THEOREM XLIX.

An Angle at the Circumference of a Circle, is measured by Half the Arc that subtends it.

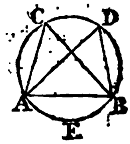


LET BAC be an angle at the circumference; it has for its measure, half the arc BC which subtends it.

For, suppose the tangent DE passing through the point of contact A. Then, the angle DAC being measured by half the arc ABC, and the angle DAB by half the arc AB (th. 48); it follows, by equal subtraction, that the difference, or angle BAC, must be measured by half the arc BC, which it stands upon. Q. E. D.

THEOREM L.

All Angles in the Same Segment of a Circle, or Standing on the Same Arc, are Equal to each other.

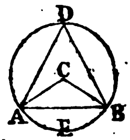


LET C and D be two angles in the same segment ACDB, or, which is the same thing, standing on the supplemental arc AEB; then will the angle C be equal to the angle D.

For each of these angles is measured by half the arc AEB; and thus, having equal measures, they are equal to each other (ax. 11).

THEOREM LI.

An Angle at the Centre of a Circle is Double the Angle at the Circumference, when both stand on the Same Arc.

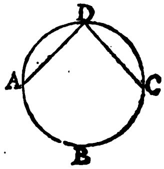


Let C be an angle at the centre C, and D an angle at the circumference, both standing on the same arc or same chord AB: then will the angle C be double of the angle D, or the angle D equal to half the angle C.

For, the angle at the centre C is measured by the whole arc AEB (def. 57), and the angle at the circumference D is measured by half the same arc AEB (th. 49); therefore the angle D is only half the angle C, or the angle C double the angle D.

THEOREM LII.

An Angle in a Semicircle, is a Right Angle.

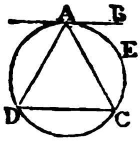


IF ABC or ADC be a Semicircle; then any angle D in that semicircle, is a right angle.

For, the angle D, at the circumference, is measured by half the arc ABC (th. 49), that is, by a quadrant of the circumference. But a quadrant is the measure of a right angle (corol. 4, th. 6; or corol. 2, th. 48). Therefore the angle D is a right angle.

THEOREM

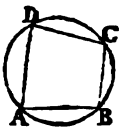
Again, the ...
to the ci... ......
(th. 41). ... to a Circle, and a Chord
CGD. Now, ... ntact, is Equal to the Angle
FG, makes the ...
equal to ... a chord, and
line ... ernate segment ADC;
angle ... be equal to the angle
equal sums ... gent and chord of the
BC, which is ...
... the circumference,



... half the arc AEC (th. 49);
... BAC, made by the tangent and chord, is also
... the same half arc AEC (th. 48); therefore these
... are equal (ax. 11).

THEOREM LIV.

... m of any Two Opposite Angles of a Quadrangle
Inscribed in a Circle, is Equal to Two Right Angles.

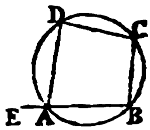


... ABCD be any quadrilateral inscribed ... ircle; then shall the sum of the two ... site angles A and C, or B and D, be ... l to two right angles.

... the angle A is measured by half the ... CB, which it stands on, and the angle ... half the arc DAB (th. 49); therefore ... sum of the two angles A and C is measured by half the ... of these two arcs, that is, by half the circumference. But half the circumference is the measure of two right angles (corol. 4, th. 6); therefore the sum of the two opposite angles A and C is equal to two right angles. In like manner it is shown, that the sum of the other two opposite ... gles, D and B, is equal to two right angles. Q. E. D.

THEOREM LV.

If any Side of a Quadrangle, Inscribed in a Circle, be Produced out, the Outward Angle will be Equal to the Inward Opposite Angle.



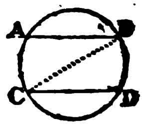
If the side AB, of the quadrilateral ... D, inscribed in a circle, be produced ... E; the outward angle DAE will be equal ... the inward opposite angle C.

For,

For, the sum of the two adjacent angles DAE and DAB is equal to two right angles (th. 6); and the sum of the two opposite angles C and DAB is also equal to two right angles (th. 54); therefore the former sum, of the two angles DAE and DAB, is equal to the latter sum, of the two C and DAB (ax. 1). From each of these equals taking away the common angle DAB, there remains the angle DAE equal the angle C. Q. E. D.

THEOREM LVI.

Any Two Parallel Chords Intercept Equal Arcs.

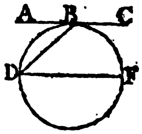


LET the two chords AB, CD, be parallel: then will the arcs AC, BD, be equal; or AC = BD.

For, draw the line BC. Then, because the lines AB, CD, are parallel, the alternate angles B and C are equal (th. 12). But the angle at the circumference B, is measured by half the arc AC (th. 49); and the other equal angle at the circumference C is measured by half the arc BD: therefore the halves of the arcs AC, BD, and consequently the arcs themselves, are also equal. Q. E. D.

THEOREM LVII.

When a Tangent and Chord are Parallel to each other, they Intercept Equal Arcs.

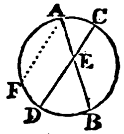


LET the tangent ABC be parallel to the chord DF; then are the arcs BD, BF, equal; that is, BD = BF.

For, draw the chord BD. Then, because the lines AB, DF, are parallel, the alternate angles D and B are equal (th. 12). But the angle B, formed by a tangent and chord, is measured by half the arc BD (th. 48); and the other angle at the circumference D is measured by half the arc BF (th. 49); therefore the arcs BD, BF, are equal. Q. E. D.

THEOREM LVIII.

The Angle formed, Within a Circle, by the Intersection of two Chords, is Measured by Half the Sum of the Two Intercepted Arcs.



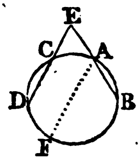
LET the two chords AB, CD, intersect at the point E: then the angle AEC, or DEB, is measured by half the sum of two arcs AC, DB.

For, draw the chord AF parallel to CD. Then, because the lines AF, CD, are parallel, and AB cuts them, the angles on the same side A and DEB are equal (th. 14). But the angle at the circumference A is measured by half the arc BF, or of the sum of FD and DB (th. 49); therefore the angle E is also measured by half the sum of FD and DB.

Again, because the chords AF, CD, are parallel, the arcs AC, FD, are equal (th. 56); therefore the sum of the two arcs AC, DB, is equal to the sum of the two FD, DB; and consequently the angle E, which is measured by half the latter sum, is also measured by half the former. Q. E. D.

THEOREM LIX.

The Angle formed, Without a Circle, by two Secants, is Measured by Half the Difference of the Intercepted Arcs.



LET the angle E be formed by two secants EAB and ECD; this angle is measured by half the difference of the two arcs AC, DB, intercepted by the two secants.

Draw the chord AF parallel to CD. Then, because the lines AF, CD, are parallel, and AB cuts them, the angles on the same side A and BED are equal (th. 14). But the angle A, at the circumference, is measured by half the arc BF (th. 49), or of the difference of DF and DB: therefore the equal angle E is also measured by half the difference of DF, DB.

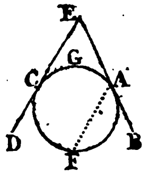
Again, because the chords AF, CD, are parallel, the arcs AC, FD, are equal (th. 56); therefore the difference of the

two arcs AC, DB, is equal to the difference of the two DF, DB. Consequently the angle E, which is measured by half the latter difference, is also measured by half the former.

Q. E. D.

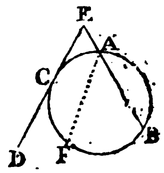
THEOREM LX.

The Angle formed by Two Tangents, is Measured by Half the Difference of the two Intercepted Arcs.



LET EB, ED, be two tangents to a circle at the points A, C; then the angle E is measured by half the difference of the two arcs CFA, CGA.

For, draw the chord AF parallel to ED. Then, because the lines AF, ED, are parallel, and EB meets them, the angles on the same side A and E are equal (th. 14). But the angle A, formed by the chord AF and tangent AB, is measured by half the arc AF (th. 48); therefore the equal angle E is also measured by half the same arc AF, or half the difference of the arcs CFA and CF, or CGA (th. 57).



Corol. In like manner it is proved, that the angle E, formed by a tangent ECD, and a secant EAB, is measured by half the difference of the two intercepted arcs CA and CFB.

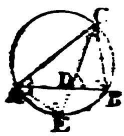
THEOREM LXI.

When two Lines, meeting a Circle each in two Points, Cut one another, either Within it or Without it: the Rectangle of the Parts of the one, is Equal to the Rectangle of the Parts of the other; the Parts of each being measured from the point of meeting to the two intersections with the circumference.

X LET

THEOREM LXIV.

The Square of a line bisecting any Angle of a Triangle, together with the Rectangle of the two Segments of the opposite Side, is Equal to the Rectangle of the two other Sides including the Bisected Angle.



LET CD bisect the angle C of the triangle ABC; then the square CD^2 + the rectangle AD . DB is = the rectangle AC . CB.

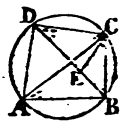
For, let CD be produced to meet the circumscribing circle at E, and join AE.

Then the two triangles ACE, BCD, are equiangular: for the angles at C are equal by supposition, and the angles B and E are equal, standing on the same arc AC (th. 50); consequently the third angles at A and D are equal (corol. 1, th. 17): also AC, CD, and CE, CB, are like or corresponding sides, being opposite to equal angles: therefore the rectangle AC . CB is = the rectangle CD . CE (th. 62). But the latter rectangle CD . CE is = CD^2 + the rectangle CD . DE (th. 30); therefore also the former rectangle AC . CB is also = CD^2 + CD . DE, or equal to CD^2 + AD . DB, since CD . DE is = AD . DB (th. 61).

Q. E. D.

THEOREM LXV.

The Rectangle of the two Diagonals of any Quadrangle Inscribed in a Circle, is equal to the sum of the two Rectangles of the Opposite Sides.



LET ABCD be any quadrilateral inscribed in a circle, and AC, BD, its two diagonals: then the rectangle AC . BD is = the rectangle AB . DC + the rectangle AD . BC.

For, let CE be drawn, making the angle BCE equal to the angle DCA. Then the two triangles ACD, BCE, are equiangular; for the angles A and B are equal, standing on the same arc DC; and the angles DCA, BCE, are equal by supposition; consequently the third angles ADC, BEC, are also equal: also, AC, BC, and AD, BE, are like or corresponding sides, being opposite to the equal angles: therefore the rectangle AC . BE is = the rectangle AD . BC (th. 62).

Again,

Again, the two triangles ABC, DEC, are equiangular: for the angles BAC, BDC, are equal, standing on the same arc BC; and the angle DCE is equal to the angle BCA, by adding the common angle ACE to the two equal angles DCA, BCE; therefore the third angles E and ABC are also equal: but AC, DC, and AB, DE, are the like sides: therefore the rectangle AC . DE is = the rectangle AB . DC (th. 62).

Hence, by equal additions, the sum of the rectangles AC . BE + AC . DE is = AD . BC + AB . DC. But the former sum of the rectangles AC . BE + AC . DE is = the rectangle AC . BD (th. 30): therefore the same rectangle AC . BD is equal to the latter sum, the rect. AD . BC + the rect. AB . DC (ax. 1). Q. E. D.

OF RATIOS AND PROPORTIONS.

DEFINITIONS.

DEF. 76. RATIO is the proportion or relation which one magnitude bears to another magnitude of the same kind, with respect to quantity.

Note. The measure, or quantity, of a ratio, is conceived, by considering what part or parts the leading quantity, called the Antecedent, is of the other, called the Consequent; or what part or parts the number expressing the quantity of the former, is of the number denoting in like manner the latter. So, the ratio of a quantity expressed by the number 2, to a like quantity expressed by the number 6, is denoted by 6 divided by 2, or $\frac{6}{2}$ or 3: the number 2 being 3 times contained in 6, or the third part of it. In like manner, the ratio of the quantity 3 to 6, is measured by $\frac{6}{3}$ or 2; the ratio of 4 to 6 is $\frac{6}{4}$ or $1\frac{1}{2}$; that of 6 to 4 is $\frac{4}{6}$ or $\frac{2}{3}$; &c.

77. Proportion is an equality of ratios. Thus,

78. Three quantities are said to be Proportional, when the ratio of the first to the second is equal to the ratio of the second to the third. As of the three quantities A (2), B (4), C (8), where $\frac{4}{2} = \frac{8}{4} = 2$, both the same ratio.

79. Four quantities are said to be Proportional, when the ratio of the first to the second, is the same as the ratio of the third to the fourth. As of the four, A (2), B (4), C (5), D (10), where $\frac{4}{2} = \frac{10}{5} = 2$, both the same ratio.

Note.

Note. To denote that four quantities, A, B, C, D, are proportional, they are usually stated or placed thus, A : B :: C : D; and read thus, A is to B as C is to D. But when three quantities are proportional, the middle one is repeated, and they are written thus, A : B :: B : C.

80. Of three proportional quantities, the middle one is said to be a Mean Proportional between the other two; and the last, a Third Proportional to the first and second.

81. Of four proportional quantities, the last is said to be a Fourth Proportional to the other three, taken in order.

82. Quantities are said to be Continually Proportional, or in Continued Proportion, when the ratio is the same between every two adjacent terms, viz. when the first is to the second, as the second to the third, as the third to the fourth, as the fourth to the fifth, and so on, all in the same common ratio. As in the quantities 1, 2, 4, 8, 16, &c; where the common ratio is equal to 2.

83. Of any number of quantities, A, B, C, D, the ratio of the first A, to the last D, is said to be Compounded of the ratios of the first to the second, of the second to the third, and so on to the last.

84. Inverse ratio is, when the antecedent is made the consequent, and the consequent the antecedent.—Thus, if 1 : 2 :: 3 : 6; then inversely, 2 : 1 :: 6 : 3.

85. Alternate proportion is, when antecedent is compared with antecedent, and consequent with consequent.—As, if 1 : 2 :: 3 : 6; then, by alternation, or permutation, it will be 1 : 3 :: 2 : 6.

86. Compounded ratio is, when the sum of the antecedent and consequent is compared, either with the consequent, or with the antecedent.—Thus, if 1 : 2 :: 3 : 6, then by composition, $1 + 2 : 1 :: 3 + 6 : 3$, and $1 + 2 : 2 :: 3 + 6 : 6$.

87. Divided ratio, is when the difference of the antecedent and consequent is compared, either with the antecedent or with the consequent.—Thus, if 1 : 2 : : 3 : 6, then, by division, $2 - 1 : 1 :: 6 - 3 : 3$, and $2 - 1 : 2 :: 6 - 3 : 6$.

Note. The term Divided, or Division, here means subtracting, or parting; being used in the sense opposed to compounding, or adding, in def. 86.

THEOREM

THEOREM LXVI.

Equimultiples of any two Quantities have the same Ratio as the Quantities themselves.

LET A and B be any two quantities, and mA, mB, any equimultiples of them, m being any number whatever: then will mA and mB have the same ratio as A and B, or $A : B :: mA : mB$.

For $\frac{mB}{mA} = \frac{B}{A}$, the same ratio.

Corol. Hence, like parts of quantities have the same ratio as the wholes; because the wholes are equimultiples of the like parts, or A and B are like parts of mA and mB.

THEOREM LXVII.

If Four Quantities, of the Same Kind, be Proportionals; they will be in Proportion by Alternation or Permutation, or the Antecedents will have the Same Ratio as the Consequents.

LET $A : B :: mA : mB$; then will $A : mA :: B : mB$.

For $\frac{mA}{A} = m$, and $\frac{mB}{B} = m$, both the same ratio.

THEOREM LXVIII.

If Four Quantities be Proportional; they will be in Proportion by Inversion, or Inversely.

LET $A : B :: mA : mB$; then will $B : A :: mB : mA$.

For $\frac{mA}{mB} = \frac{A}{B}$, both the same ratio.

THEOREM LXIX.

If Four Quantities be Proportional; they will be in Proportion by Composition and Division.

LET $A : B :: mA : mB$;
Then will $B \pm A : A :: mB \pm mA : mA$,
and $B \pm A : B :: mB \pm mA : mB$.

For, $\frac{mA}{mB \pm mA} = \frac{A}{B \pm A}$; and $\frac{mB}{mB \pm mA} = \frac{B}{B \pm A}$.

Corol.

Corol. It appears from hence, that the Sum of the Greatest and Least of four proportional quantities, of the same kind, exceeds the Sum of the Two Means. For, since - - - - $A : A + B :: mA : mA + mB$, where A is the least, and $mA + mB$ the greatest; then $\overline{m+1} . A + mB$, the sum of the greatest and least, exceeds $\overline{m+1} . A + B$ the sum of the two means.

THEOREM LXX.

If, of Four Proportional Quantities, there be taken any Equimultiples whatever of the two Antecedents, and any Equimultiples whatever of the two Consequents; the quantities resulting will still be proportional.

LET $A : B :: mA : mB$; also, let pA and pmA be any equimultiples of the two antecedents, and qB and qmB any equimultiples of the two consequents; then will - - - - - $pA : qB :: pmA : qmB$.

For $\frac{qmB}{pmA} = \frac{qB}{pA}$, both the same ratio.

THEOREM LXXI.

If there be Four Proportional Quantities, and the two Consequents be either Augmented or Diminished by Quantities that have the Same Ratio as the respective Antecedents; the Results and the Antecedents will still be Proportionals.

LET $A : B :: mA : mB$, and nA and nmA any two quantities having the same ratio as the two antecedents; then will $A : B \pm nA :: mA : mB \pm nmA$.

For $\frac{mB \pm nmA}{mA} = \frac{B \pm nA}{A}$, both the same ratio.

THEOREM LXXII.

If any Number of Quantities be Proportional, then any one of the Antecedents will be to its Consequent, as the Sum of all the Antecedents is to the Sum of all the Consequents.

LET $A : B :: mA : mB :: nA : nB$, &c; then will - - - - $A : B :: A + mA + nA :: B + mB + nB$, &c.

For $\frac{B + mB + nB}{A + mA + nA} = \frac{B}{A}$, the same ratio.

THEOREM

THEOREM LXXIII.

If a Whole Magnitude be to a Whole, as a Part taken from the first, is to a Part taken from the other; then the Remainder will be to the Remainder, as the whole to the whole.

LET $A : B :: \frac{m}{n} A : \frac{m}{n} B$;

then will $A : B :: A - \frac{m}{n} A : B - \frac{m}{n} B$.

For $\frac{B - \frac{m}{n} B}{A - \frac{m}{n} A} = \frac{B}{A}$, both the same ratio.

THEOREM LXXIV.

If any Quantities be Proportional; their Squares, or Cubes, or any Like Powers, or Roots, of them, will also be Proportional.

LET $A : B :: mA : mB$; then will $A^n : B^n :: m^n A^n : m^n B^n$.

For $\frac{m^n B^n}{m^n A^n} = \frac{B^n}{A^n}$, both the same ratio.

THEOREM LXXV.

If there be two Sets of Proportionals; then the Products or Rectangles of the Corresponding Terms will also be Proportional.

LET $A : B :: mA : mB$,
and $C : D :: nC : nD$;
then will $AC : BD :: mnAC : mnBD$.

For $\frac{mnBD}{mnAC} = \frac{BD}{AC}$, both the same ratio.

THEOREM LXXVI.

If Four Quantities be Proportional; the Rectangle or Product of the two Extremes, will be Equal to the Rectangle or Product of the two Means. And the converse.

LET $A : B :: mA : mB$;
then is $A \times mB = B \times mA = mAB$, as is evident.

THEOREM

THEOREM LXXVII.

If Three Quantities be Continued Proportionals; the Rectangle or Product of the two Extremes, will be Equal to the Square of the Mean. And the converse.

LET A, mA, m^2A be three proportionals,
or A : mA : : mA : m^2A;
then is A × m^2A = m^2A^2, as is evident.

THEOREM LXXVIII.

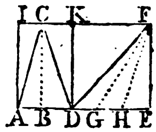
If any Number of Quantities be Continued Proportionals; the Ratio of the First to the Third, will be duplicate or the Square of the Ratio of the First and Second; and the Ratio of the First and Fourth will be triplicate or the cube of that of the First and Second; and so on.

LET A, mA, m^2A, m^3A, &c, be proportionals;

then is $\frac{m\text{A}}{\text{A}} = m$; but $\frac{m^2\text{A}}{\text{A}} = m^2$; and $\frac{m^3\text{A}}{\text{A}} = m^3$; &c.

THEOREM LXXIX.

Triangles, and also Parallelograms, having equal Altitudes, are to each other as their Bases.



LET the two triangles ADC, DEF, have the same altitude, or be between the same parallels AE, CF; then is the surface of the triangle ADC, to the surface of the triangle DEF, as the base AD is to the base DE. Or, AD : DE : : the triangle ADC : the triangle DEF.

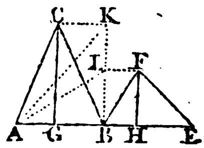
For, let the base AD be to the base DE, as any one number m (2), to any other number n (3); and divide the respective bases into those parts, AB, BD, DG, GH, HE, all equal to one another; and from the points of division draw the lines BC, FG, FH, to the vertices C and F. Then will these lines divide the triangles ADC, DEF, into the same number of parts as their bases, each equal to the triangle ABC, because those triangular parts have equal bases and altitude (corol. 2, th. 25); namely, the triangle ABC, equal to each of the triangles BDC, DFG, GFH, HFE. So that the triangle ADC, is to the triangle DFE, as the number of parts

parts *m* (2) of the former, to the number *n* (3) of the latter, that is, as the base AD to the base DE (def. 79).

In like manner, the parallelogram ADKI is to the parallelogram DEFK, as the base AD is to the base DE; each of these having the same ratio as the number of their parts, *m* to *n*. Q. E. D.

THEOREM LXXX.

Triangles, and also Parallelograms, having Equal Bases, are to each other as their Altitudes.



LET ABC, BEF, be two triangles having the equal bases AB, BE, and whose altitudes are the perpendiculars CG, FH; then will the triangle ABC : the triangle BEF :: CG : FH.

For, let BK be perpendicular to AB, and equal to CG; in which let there be taken BL = FH; drawing AK and AL.

Then, triangles of equal bases and heights being equal (corol. 2, th. 25), the triangle ABK is = ABC, and the triangle ABL = BEF. But, considering now ABK, ABL, as two triangles on the bases BK, BL, and having the same altitude AB, these will be as their bases (th. 79), namely, the triangle ABK : the triangle ABL :: BK : BL.

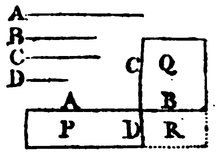
But the triangle ABK = ABC, and the triangle ABL = BEF,
also BK = CG, and BL = FH.
Theref. the triangle ABC : triangle BEF :: CG : FH.

And since parallelograms are the doubles of these triangles, having the same bases and altitudes, they will likewise have to each other the same ratio as their altitudes. Q. E. D.

Corol. Since, by this theorem, triangles and parallelograms, when their bases are equal, are to each other as their altitudes; and by the foregoing one, when their altitudes are equal, they are to each other as their bases; therefore universally, when neither are equal, they are to each other in the compound ratio, or as the rectangle or product of their bases and altitudes.

THEOREM LXXXI.

If Four Lines be Proportional; the Rectangle of the Extremes will be Equal to the Rectangle of the Means. And, conversely, if the Rectangle of the Extremes, of four Lines, be equal to the Rectangle of the Means, the Four Lines, taken alternately, will be Proportional.



LET the four lines A, B, C, D, be proportionals, or A : B :: C : D; then will the rectangle of A and D be equal to the rectangle of B and C; or the rectangle A . D = B . C.

For, let the four lines be placed with their four extremities meeting in a common point, forming at that point four right angles; and draw lines parallel to them to complete the rectangles P, Q, R, where P is the rectangle of A and D, Q the rectangle of B and C, and R the rectangle of B and D.

Then the rectangles P and R, being between the same parallels, are to each other as their bases A and B (th. 79); and the rectangles Q and R, being between the same parallels, are to each other as their bases C and D. But the ratio of A to B, is the same as the ratio of C to D, by hypothesis; therefore the ratio of P to R, is the same as the ratio of Q to R; and consequently the rectangles P and Q are equal. Q. E. D.

Again, if the rectangle of A and D, be equal to the rectangle of B and C; these lines will be proportional, or A : B :: C : D.

For, the rectangles being placed the same as before: then, because parallelograms between the same parallels, are to one another as their bases, the rectangle P : R :: A : B, and Q : R :: C : D. But as P and Q are equal, by supposition, they have the same ratio to R, that is, the ratio of A to B is equal to the ratio of C to D, or A : B :: C : D. Q. E. D.

Corol. 1. When the two means, namely, the second and third terms, are equal, their rectangle becomes a square of the second term, which supplies the place of both the second and third. And hence it follows, that when three lines are proportionals, the rectangle of the two extremes is equal to the

the square of the mean; and, conversely, if the rectangle of the extremes be equal to the square of the mean, the three lines are proportionals.

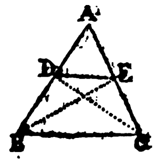
Corol. 2. Since it appears, by the rules of proportion in Arithmetic and Algebra, that when four quantities are proportional, the product of the extremes is equal to the product of the two means; and, by this theorem, the rectangle of the extremes is equal to the rectangle of the two means; it follows, that the area or space of a rectangle is represented or expressed by the product of its length and breadth multiplied together. And, in general, a rectangle in geometry is similar to the product of the measures of its two dimensions of length and breadth, or base and height. Also, a square is similar to, or represented by, the measure of its side multiplied by itself. So that, what is shown of such products, is to be understood of the squares and rectangles.

Corol. 3. Since the same reasoning, as in this theorem, holds for any parallelograms whatever, as well as for the rectangles, the same property belongs to all kinds of parallelograms, having equal angles, and also to triangles, which are the halves of parallelograms; namely, that if the sides about the equal angles of parallelograms, or triangles, be reciprocally proportional, the parallelograms or triangles will be equal; and, conversely, if the parallelograms or triangles be equal, their sides about the equal angles will be reciprocally proportional.

Corol. 4. Parallelograms, or triangles, having an angle in each equal, are in proportion to each other as the rectangles of the sides which are about these equal angles.

THEOREM LXXXII.

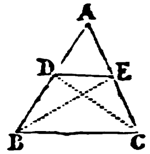
If a Line be drawn in a Triangle Parallel to one of its sides, it will cut the two other Sides Proportionally.



LET DE be parallel to the side BC of the triangle ABC; then will AD : DB :: AE : EC.

For, draw BE and CD. Then the triangles DBE, DCE, are equal to each other, because they have the same base DE, and are between the same parallels DE, BC (th. 25). But the two triangles ADE, BDE, on the bases AD, DB, have the same alti-

tude; and the two triangles ADE, CDE, on the bases AE, EC, have also the same altitude; and because triangles of the same altitude are to each other as their bases, therefore



the triangle ADE : BDE :: AD : DB,
and triangle ADE : CDE :: AE : EC.

But BDE is = CDE; and equals must have to equals the same ratio; therefore AD : DB :: AE : EC. Q. E. D.

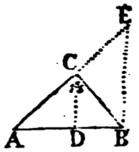
Corol. Hence, also, the whole lines AB, AC, are proportional to their corresponding proportional segments (corol. th. 66),

viz. AB : AC :: AD : AE,
and AB : AC :: BD : CE.

THEOREM LXXXIII.

A Line which Bisects any Angle of a Triangle, divides the opposite Side into Two Segments, which are Proportional to the two other Adjacent Sides.

Let the angle ACB, of the triangle ABC, be bisected by the line CD, making the angle *r* equal to the angle *s*: then will the segment AD be to the segment DB, as the side AC is to the side CB. Or, - - - - AD : DB :: AC : CB.

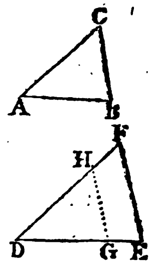


For, let BE be parallel to CD, meeting AC produced at E. Then, because the line BC cuts the two parallels CD, BE, it makes the angle CBE equal to the alternate angle *s* (th. 12), and therefore also equal to the angle *r*, which is equal to *s* by the supposition. Again, because the line AE cuts the two parallels DC, BE, it makes the angle E equal to the angle *r* on the same side of it (th. 14). Hence, in the triangle BCE, the angles B and E, being each equal to the angle *r*, are equal to each other, and consequently their opposite sides CB, CE, are also equal (th. 3).

But now, in the triangle ABE, the line CD, being drawn parallel to the side BE, cuts the two other sides AB, AE, proportionally (th. 82), making AD to DB, as is AC to CE or to [e]qual CB. Q. E. D.

THEOREM LXXXIV.

Equiangular Triangles are Similar, or have their Like Sides Proportional.



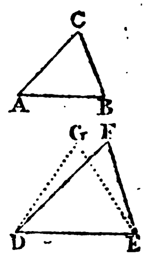
LET ABC, DEF, be two equiangular triangles, having the angle A equal to the angle D, the angle B to the angle E, and consequently the angle C to the angle F; then will AB : AC : : DE : DF.

For, make DG = AB, and DH = AC, and join GH. Then the two triangles ABC, DGH, having the two sides AB, AC, equal to the two DG, DH, and the contained angles A and D also equal, are identical, or equal in all respects (th. 1), namely, the angles B and C are equal to the angles G and H. But the angles B and C are equal to the angles E and F by the hypothesis; therefore also the angles G and H are equal to the angles E and F (ax. 1), and consequently the line GH is parallel to the side EF (cor. 1, th. 14).

Hence then, in the triangle DEF, the line GH, being parallel to the side EF, divides the two other sides proportionally, making DG : DH : : DE : DF (cor. th. 82). But DG and DH are equal to AB and AC; therefore also - - - - AB : AC : : DE : DF. Q. E. D.

THEOREM LXXXV.

Triangles which have their Sides Proportional, are Equiangular.



IN the two triangles ABC, DEF, if AB : DE : : AC : DF : : BC : EF; the two triangles will have their corresponding angles equal.

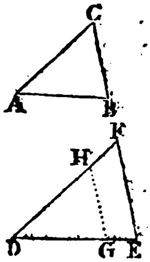
For, if the triangle ABC be not equiangular with the triangle DEF, suppose some other triangle, as DEG, to be equiangular with ABC. But this is impossible: for if the two triangles ABC, DEG, were equiangular, their sides would be proportional (th. 84). So that, AB being to DE as AC to DG, and AB to DE as BC to EG, it follows that DG and EG, being fourth proportionals to the same three quantities,

as

as well as the two DF, EF, the former DG, EG, would be equal to the latter, DF, EF. Thus then, the two triangles DEF, DEG, having their three sides equal, would be identical (th. 5); which is absurd, since their angles are unequal.

THEOREM LXXXVI.

Triangles, which have an Angle in the one Equal to an Angle in the other, and the Sides about these angles Proportional, are Equiangular.



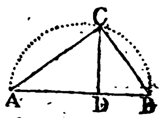
LET ABC, DEF, be two triangles, having the angle A = the angle D, and the sides AB, AC, proportional to the sides DE, DF: then will the triangle ABC be equiangular with the triangle DEF.

For, make DG = AB, and DH = AC, and join GH.

Then, the two triangles ABC, DGH, having two sides equal, and the contained angles A and D equal, are identical and equiangular (th. 1), having the angles G and H equal to the angles B and C. But, since the sides DG, DH, are proportional to the sides DE, DF, the line GH is parallel to EF (th. 82); hence the angles E and F are equal to the angles G and H (th. 14), and consequently to their equals B and C. Q. E. D.

THEOREM LXXXVII.

In a Right-Angled Triangle, a Perpendicular from the Right Angle, is a Mean Proportional between the Segments of the Hypothenuse; and each of the Sides, about the Right Angle, is a Mean Proportional between the Hypothenuse and the adjacent segment.



LET ABC be a right-angled triangle, and CD a perpendicular from the right angle C to the hypothenuse AB; then will

CD be a mean proportional between AD and DB;
AC a mean proportional between AB and AD;
BC a mean proportional between AB and BD.

Or, AD : CD :: CD : DB; and AB : BC :: BC : BD; and AB : AC :: AC : AD.

For,

For, the two triangles ABC, ADC, having the right angles at C and D equal, and the angle A common, have their third angles equal, and are equiangular (cor. 1, th. 17). In like manner, the two triangles ABC, BDC, having the right angles at C and D equal, and the angle B common, have their third angles equal, and are equiangular.

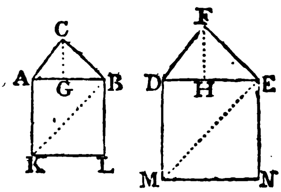
Hence then, all the three triangles ABC, ADC, BDC, being equiangular, will have their like sides proportional (th. 84);

viz. AD : CD :: CD : DB;
and AE : AC :: AC : AD;
and AB : BC :: BC : BD. Q. E. D.

Corol. Because the angle in a semicircle is a right angle (th. 52); it follows, that if, from any point C in the periphery of the semicircle, a perpendicular be drawn to the diameter AB; and the two chords CA, CB, be drawn to the extremities of the diameter: then are AC, BC, CD, the mean proportionals as in this theorem, or (by th. 77), - - $CD^2 = AD \cdot DB$; $AC^2 = AB \cdot AD$; and $BC^2 = AB \cdot BD$.

THEOREM LXXXVIII.

Equiangular or Similar Triangles, are to each other as the Squares of their Like Sides.



LET ABC, DEF, be two equiangular triangles, AB and DE being two like sides: then will the triangle ABC be to the triangle DEF, as the square of AB is to the square of DE, or as AB^2 to DE^2.

For, let AL and DN be the squares on AB and DE; also draw their diagonals BK, EM, and the perpendiculars CG, FH, of the two triangles.

Then, since equiangular triangles have their like sides proportional (th. 84), in the two equiangular triangles ABC, DEF, the side AC : DF :: AB : DE; and in the two ACG, DFH, the side AC : DF :: CG : FH; therefore, by equality CG : FH :: AB : DE, or CG : AB :: FH : DE.

But because triangles on equal bases are to each other as their altitudes, the triangles ABC, ABK, on the same base AB, are to each other, as their altitudes CG, AK, or AB: and

and the triangles DEF, DEM, on the same base DE, are as their altitudes FH, DM, or DE;

that is, triangle ABC : triangle ABK :: CG : AB,
and triangle DEF : triangle DEM :: FH : DE.

But it has been shown that CG : AB :: FH : DE;
theref. of equality $\triangle$ABC : $\triangle$ABK :: $\triangle$DEF : $\triangle$DEM,
or alternately, as $\triangle$ABC : $\triangle$DEF :: $\triangle$ABK : $\triangle$DEM.

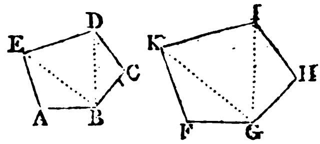
But the squares AL, DN, being the double of the triangles ABK, DEM, have the same ratio with them;

therefore the $\triangle$ABC : $\triangle$DEF :: square AL : square DN.

Q. E. D.

THEOREM LXXXIX.

All Similar Figures are to each other, as the Squares of their Like Sides.



Let ABCDE, FGHIK, be any two similar figures, the like sides being AB, FG, and BC, GH, and so on in the same order: then will the figure ABCDE be to the figure FGHIK, as the square of AB to the square of FG, or as AB^2 to FG^2.

For, draw BE, BD, GK, GI, dividing the figures into an equal number of triangles, by lines from two equal angles B and G.

The two figures being similar (by suppos.), they are equiangular, and have their like sides proportional (def. 67).

Then, since the angle A is = the angle F, and the sides AB, AE, proportional to the sides FG, FK, the triangles ABE, FGK, are equiangular (th. 86). In like manner, the two triangles BCD, GHI, having the angle C = the angle H, and the sides BC, CD, proportional to the sides GH, HI, are also equiangular. Also, if from the equal angles AED, FKI, there be taken the equal angles AEB, FKG, there will remain the equals BED, GKI; and if from the equal angles CDE, HIK, be taken away the equals CDB, HIG, there will remain the equals BDE, GIK; so that the two triangles BDE, GIK, having two angles equal, are also equiangular. Hence each triangle of the one figure, is equiangular with each corresponding triangle of the other.

But equiangular triangles are similar, and are proportional to the squares of their like sides (th. 88).

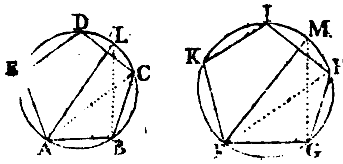
Therefore

Therefore the $\triangle$ ABE : $\triangle$ FGK :: $AB^2 : FG^2$,
and $\triangle$ BCD : $\triangle$ GHI :: $BC^2 : GH^2$,
and $\triangle$ BDE : $\triangle$ GIK :: $DE^2 : IK^2$.

But as the two polygons are similar, their like sides are proportional, and consequently their squares also proportional; so that all the ratios AB^2 to FG^2, and BC^2 to GH^2, and DE^2 to IK^2, are equal among themselves, and consequently the corresponding triangles also, ABE to FGK, and BCD to GHI, and BDE to GIK, have all the same ratio, viz. that of AB^2 to FG^2: and hence all the antecedents, or the figure ABCDE, have to all the consequents, or the figure FGHIK, still the same ratio, viz. that of AB^2 to FG^2 (th. 72). Q. E. D.

THEOREM XC.

Similar Figures Inscribed in Circles, have their Like Sides, and also their Whole Perimeters, in the Same Ratio as the Diameters of the Circles in which they are Inscribed.



LET ABCDE, FGHIK, be two similar figures, inscribed in the circles whose diameters are AL and FM; then will each side AB, BC, &c, of the one figure be to the like side GF, GH, &c, of the other figure, or the whole perimeter AB + BC + &c, of the one figure, to the whole perimeter FG + GH + &c, of the other figure, as the diameter AL to the diameter FM.

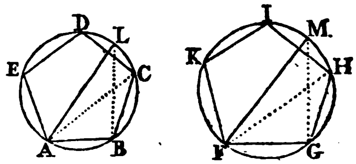
For, draw the two corresponding diagonals AC, FH, as also the lines BL, GM. Then, since the polygons are similar, they are equiangular, and their like sides have the same ratio (def. 67); therefore the two triangles ABC, FGH, have the angle B = the angle G, and the sides AB, BC, proportional to the two sides FG, GH, consequently these two triangles are equiangular (th. 86), and have the angle ACB = FHG. But the angle ACB = ALB, standing on the same arc AB; and the angle FHG = FMG, standing on the same arc FG; therefore the angle ALB = FMG (ax. 1). And since the angle ABL = FGM, being both right angles, because in a semicircle; therefore the two triangles ABL, FGM, having two angles equal, are equiangular; and consequently their like

like sides are proportional (th. 84); hence AB : FG :: the diameter AL : the diameter FM.

In like manner, each side BC, CD, &c, has to each side GH, HI, &c, the same ratio of AL to FM; and consequently the sums of them are still in the same ratio; viz, AB + BC + CD, &c : FG + GH + HI, &c :: the diam. AL : the diam. FM (th. 72). Q. E. D.

THEOREM XCI.

Similar Figures Inscribed in Circles, are to each other as the Squares of the Diameters of those Circles.



LET ABCDE, FGHIK, be two similar figures, inscribed in the circles whose diameters are AL and FM; then the surface of the polygon ABCDE will be to the surface of the polygon FGHIK, as AL^2 to FM^2.

For, the figures being similar, are to each other as the squares of their like sides, AB^2 to FG^2 (th. 88). But, by the last theorem, the sides AB, FG, are as the diameters AL, FM; and therefore the squares of the sides AB^2 to FG^2, as the squares of the diameters AL^2 to FM^2 (th. 74). Consequently the polygons ABCDE, FGHIK, are also to each other as the squares of the diameters AL^2 to FM^2 (ax. 1). Q. E. D.

THEOREM XCII.

The Circumferences of all Circles are to each other as their Diameters.

LET D, *d*, denote the diameters of two circles, and C, *c*, their circumferences;

then will D : *d* :: C : *c*, or D : C :: *d* : *c*.

For (by theor. 90), similar polygons inscribed in circles have their perimeters in the same ratio as the diameters of those circles.

Now, as this property belongs to all polygons, whatever the number of the sides may be; conceive the number of the sides to be indefinitely great, and the length of each indefinitely small, till they coincide with the circumference of

the

the circle, and be equal to it, indefinitely near. Then the perimeter of the polygon of an infinite number of sides, is the same thing as the circumference of the circle. Hence it appears that the circumferences of the circles, being the same as the perimeters of such polygons, are to each other in the same ratio as the diameters of the circles. Q. E. D.

THEOREM XCIII.

The Areas or Spaces of Circles, are to each other as the Squares of their Diameters, or of their Radii.

LET A, a, denote the areas or spaces of two circles, and D, d, their diameters; then $A : a :: D^2 : d^2$.

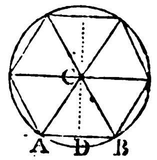
For (by theorem 91) similar polygons inscribed in circles are to each other as the squares of the diameters of the circles.

Hence, conceiving the number of the sides of the polygons to be increased more and more, or the length of the sides to become less and less, the polygon approaches nearer and nearer to the circle, till at length, by an infinite approach, they coincide, and become in effect equal; and then it follows, that the spaces of the circles, which are the same as of the polygons, will be to each other as the squares of the diameters of the circles. Q. E. D.

Corol. The spaces of circles are also to each other as the squares of the circumferences; since the circumferences are in the same ratio as the diameters (by theorem 92).

THEOREM XCIV.

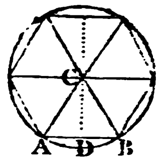
The Area of any Circle, is Equal to the Rectangle of Half its Circumference and Half its Diameter.



CONCEIVE a regular polygon to be inscribed in the circle; and radii drawn to all the angular points, dividing it into as many equal triangles as the polygon has sides, one of which is ABC, of which the altitude is the perpendicular CD from the centre to the base AB.

Then the triangle ABC, being equal to a rectangle of half the base and equal altitude (th. 26, cor. 2), is equal to the rectangle of the half base AD and the altitude CD; conse-

consequently the whole polygon, or all the triangles added together which compose it, is equal to the rectangle of the common altitude CD, and the halves of all the sides, or the half perimeter of the polygon.



Now, conceive the number of sides of the polygon to be indefinitely increased; then will its perimeter coincide with the circumference of the circle, and consequently the altitude CD will become equal to the radius, and the whole polygon equal to the circle. Consequently the space of the circle, or of the polygon in that state, is equal to the rectangle of the radius and half the circumference. Q. E. D.

OF PLANES AND SOLIDS.

DEFINITIONS.

DEF. 88. The Common Section of two Planes, is the line in which they meet, to cut each other.

89. A Line is Perpendicular to a Plane, when it is perpendicular to every line in that plane which meets it.

90. One Plane is Perpendicular to Another, when every line of the one, which is perpendicular to the line of their common section, is perpendicular to the other.

91. The Inclination of one Plane to another, or the angle they form between them, is the angle contained by two lines, drawn from any point in the common section, and at right angles to the same, one of these lines in each plane.

92. Parallel Planes, are such as being produced ever so far both ways, will never meet, or which are every where at an equal perpendicular distance.

93. A Solid Angle, is that which is made by three or e plane angles, meeting each other in the same point.

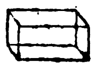
94. Similar Solids, contained by plane figures, are such as have all their solid angles equal, each to each, and are bounded by the same number of similar planes, alike placed.

95. A Prism, is a solid whose ends are parallel, equal, and like plane figures, and its sides, connecting those ends, are parallelograms.

96. A Prism takes particular names according to the figure of its base or ends, whether triangular, square, rectangular, pentagonal, hexagonal, &c.

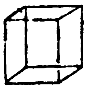
97. A Right or Upright Prism, is that which has the planes of the sides perpendicular to the planes of the ends or base.

98. A Parallelopiped, or Parallelopipedon, is a prism bounded by six parallelograms, every opposite two of which are equal, alike, and parallel.

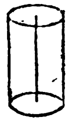


99. A Rectangular Parallelopipedon, is that whose bounding planes are all rectangles, which are perpendicular to each other.

100. A Cube, is a square prism, being bounded by six equal square sides or faces, and are perpendicular to each other.

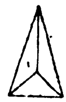


101. A Cylinder is a round prism, having circles for its ends; and is conceived to be formed by the rotation of a right line about the circumferences of two equal and parallel circles, always parallel to the axis.



102. The Axis of a Cylinder, is the right line joining the centres of the two parallel circles, about which the figure is described.

103. A Pyramid, is a solid, whose base is any right-lined plane figure, and its sides triangles, having all their vertices meeting together in a point above the base, called the Vertex of the pyramid.



104. A pyramid, like the prism, takes particular names from the figure of the base.

105. A Cone, is a round pyramid, having a circular base, and is conceived to be generated by the rotation of a right line about the circumference of a circle, one end of which is fixed at a point above the plane of that circle.

106. The

106. The Axis of a cone, is the right line, joining the vertex, or fixed point, and the centre of the circle about which the figure is described.

107. Similar Cones and Cylinders, are such as have their altitudes and the diameters of their bases proportional.

108. A Sphere, is a solid bounded by one curve surface, which is every where equally distant from a certain point within, called the Centre. It is conceived to be generated by the rotation of a semicircle about its diameter, which remains fixed.

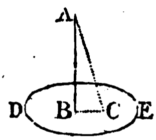
109. The Axis of a Sphere, is the right line about which the semicircle revolves; and the centre is the same as that of the revolving semicircle.

110. The Diameter of a Sphere, is any right line passing through the centre, and terminated both ways by the surface.

111. The Altitude of a Solid, is the perpendicular drawn from the vertex to the opposite side or base.

THEOREM XCV.

A Perpendicular is the Shortest Line which can be drawn from any Point to a Plane.



Let AB be perpendicular to the plane DE; then any other line, as AC, drawn from the same point A to the plane, will be longer than the line AB.

In the plane draw the line BC, joining the points B, C.

Then, because the line AB is perpendicular to the plane DE, the angle B is a right angle (def. 90), and consequently greater than the angle C; therefore the line AB, opposite to the less angle, is less than any other line AC, opposite the greater angle (th. 21). Q. E. D.

THEOREM XCVI.

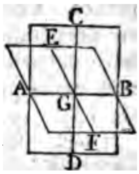
A Perpendicular Measures the Distance of any Point from a Plane.

The distance of one point from another is measured by a right line joining them, because this is the shortest line which can be drawn from one point to another. So, also, the distance from a point to a line, is measured by a perpendicular, because this line is the shortest which can be drawn from

from the point to the line. In like manner, the distance from a point to a plane, must be measured by a perpendicular drawn from that point to the plane, because this is the shortest line which can be drawn from the point to the plane.

THEOREM XCVII.

The Common Section of Two Planes, is a Right Line.



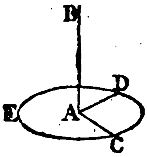
LET ACBDA, AEBFA, be two planes cutting each other; and A, B, two points in which the two planes meet; drawing the line AB, this line will be the common intersection of the two planes.

For, because the right line AB touches the two planes in the points A and B, it touches them in all other points (def. 20): this line is therefore common to the two planes: That is, the common intersection of the two planes is a right line.

Q. E. D.

THEOREM XCVIII.

If a Line be Perpendicular to two other Lines, at their Common Point of Meeting; it will be Perpendicular to the Plane of those Lines.

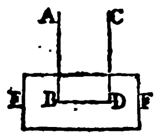


LET the line AB make right angles with the lines AC, AD; then will it be perpendicular to the plane CDE which passes through these lines.

If the line AB were not perpendicular to the plane CDE, another plane might pass through the point A, to which the line AB would be perpendicular. But this is impossible; for, since the angles BAC, BAD, are right angles, this other plane must pass through the points C, D. Hence, this plane passing through the two points A, C, of the line AC, and through the two points A, D, of the line AD, it will pass through both these two lines, and therefore be the same plane with the former. Q. E. D.

THEOREM XCIX.

If Two Lines be Perpendicular to the Same Plane, they will be Parallel to each other.



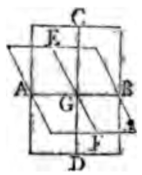
LET the two lines AB, CD, be both perpendicular to the same plane EBDF; then will AB be paralled to CD.

For, join B, D, by the line BD in the plane. Then, because the lines AB, CD, are perpendicular to the plane EF, they are both perpendicular to the line BD (def. 90) in that plane; and consequently they are parallel to each other (corol. th. 13). Q. E. D.

Corol. If two lines be parallel, and if one of them be perpendicular to any plane, the other will also be perpendicular to the same plane.

THEOREM C,

If Two Planes Cut each other at Right Angles, and a Line be drawn in one of the Planes Perpendicular to their Common Intersection, it will be Perpendicular to the other Plane.



LET the two planes ACBD, AEBF, cut each other at right angles; and the line CG be perpendicular to their common section AB; then will CG be also perpendicular to the other plane AEBF.

For, draw EG perpendicular to AB. Then, because the two lines GC, GE, are perpendicular to the common intersection AB, the angle CGE is the angle of inclination of the two planes (def. 92). But since the two planes cut each other perpendicularly, the angle of inclination CGE is a right angle. And since the line CG is perpendicular to the two lines GA, GE, in the plane AEBF, it is therefore perpendicular to that plane (th. 98). Q. E. D.

THEOREM

THEOREM CI.

If one Plane Meet another Plane, it will make Angles with that other Plane, which are together equal to two Right Angles.

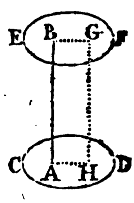
LET the plane ACBD meet the plane AEBF; these planes make with each other two angles whose sum is equal to two right angles.

For, through any point G, in the common section AB, draw CD, EF, perpendicular to AB. Then, the line CG makes with EF two angles together equal to two right angles. But these two angles are (by def. 92) the angles of inclination of the two planes. Therefore the two planes make angles with each other, which are together equal to two right angles.

Corol. In like manner, it may be demonstrated, that planes which intersect, have their vertical or opposite angles equal; also, that parallel planes have their alternate angles equal; and so on, as in parallel lines.

THEOREM CII.

If Two Planes be Parallel to each other; a Line which is Perpendicular to one of the Planes, will also be Perpendicular to the other.



LET the two planes CD, EF, be parallel, and let the line AB be perpendicular to the plane CD; then shall it also be perpendicular to the other plane EF.

For, from any point G, in the plane EF, draw GH perpendicular to the plane CD, and draw AH, BG.

Then, because BA, GH, are both perpendicular to the plane CD, the angles A and H are both right angles. And because the planes CD, EF, are parallel, the perpendiculars BA, GH, are equal (def. 93). Hence it follows that the lines BG, AH, are parallel (def. 9). And the line AB being perpendicular to the line AH, is also perpendicular to the parallel line BG (cor. th. 12).

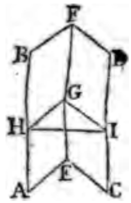
In like manner it is proved, that the line AB is perpendicular to all other lines which can be drawn from the point B in

the plane EF. Therefore the line AB is perpendicular to the whole plane EF (def. 90). Q. E. D.

THEOREM CIII.

If Two Lines be Parallel to a Third Line, though not in the same Plane with it; they will be Parallel to each other.

LET the lines AB, CD, be each of them parallel to the third line EF, though not in the same plane with it; then will AB be parallel to CD.



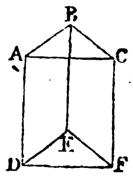
For, from any point G in the line EF, let GH, GI, be each perpendicular to EF, in the planes EB, ED, of the proposed parallels.

Then, since the line EF is perpendicular to the two lines GH, GI, it is perpendicular to the plane GHI of those lines (th. 98). And because EF is perpendicular to the plane GHI, its parallel AB is also perpendicular to that plane (cor. th. 99). For the same reason, the line CD is perpendicular to the same plane GHI. Hence, because the two lines AB, CD, are perpendicular to the same plane, these two lines are parallel (th. 99). Q. E. D.

THEOREM CIV.

If Two Lines, that meet each other, be Parallel to Two other Lines that meet each other, though not in the same Plane with them; the Angles contained by those Lines will be equal.

LET the two lines AB, BC, be parallel to the two lines DE, EF; then will the angle ABC be equal to the angle DEF.



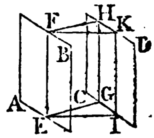
For, make the lines AB, BC, DE, EF, all equal to each other, and join AC, DF, AD, BE, CF.

Then, the lines AD, BE, joining the equal and parallel lines AB, DE, are equal and parallel (th. 24). For the same reason, CF, BE, are equal and parallel. Therefore AD, CF, are equal and parallel (th. 15); and consequently also AC, DF (th. 24). Hence, the two triangles ABC, DEF, having all their sides equal, each

each to each, have their angles also equal, and consequently the angle ABC = the angle DEF. Q. E. D.

THEOREM CV.

The Sections made by a Plane cutting two other Parallel Planes, are also Parallel to each other.



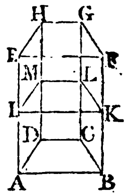
LET the two parallel planes AB, CD, be cut by the third plane EFHG, in the lines EF, GH: these two sections EF, GH, will be parallel.

Suppose EG, FH, be drawn parallel to each other in the plane EFHG; also let EI, FK, be perpendicular to the plane CD; and let IG, KH, be joined.

Then EG, FH, being parallels, and EI, FK, being both perpendicular to the plane CD, are also parallel to each other (th. 99); consequently the angle HFK is equal to the angle GEI (th. 104). But the angle FKH is also equal to the angle EIG, being both right angles; therefore the two triangles are equiangular (cor. 1 th. 17); and the sides FK, EI, being the equal distances between the parallel planes (def. 93), it follows that the sides FH, EG, are also equal (th. 2). But these two lines are parallel (by suppos.), as well as equal; consequently the two lines EF, GH, joining those equal parallels, are also parallel (th. 24). Q. E. D.

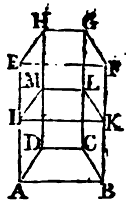
THEOREM CVI.

If any Prism be cut by a Plane Parallel to its Base, the Section will be Equal and Like to the Base.



LET AG be any prism, and IL a plane parallel to the base AC; then will the plane IL be equal and like to the base AC, or the two planes will have all their sides and all their angles equal.

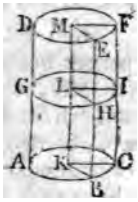
For, the two planes AC, IL, being parallel, by hypothesis; and two parallel planes, cut by a third plane, having parallel sections (th. 105); therefore IK is parallel to AB, and KL to BC, and LM to CD, and IM to AD. But AI and BK are parallels (by def. 95); consequently AK is a parallelogram; and the opposite sides AB, IK, are equal (th. 22). In like manner, it



it is shown that KL is = BC, and LM = CD, and IM = AD, or the two planes AC, IL, are mutually equilateral. But these two planes, having their corresponding sides parallel, have the angles contained by them also equal (th. 104), namely, the angle A = the angle I, the angle B = the angle K, the angle C = the angle L, and the angle D = the angle M. So that the two planes AC, IL, have all their corresponding sides and angles equal, or they are equal and like. Q. E. D.

THEOREM CVII.

If a Cylinder be cut by a Plane Parallel to its Base, the Section will be a Circle, Equal to the Base.



LET AF be a cylinder, and GHI any section parallel to the base ABC; then will GHI be a circle, equal to ABC.

For, let the planes KE, KF, pass through the axis of the cylinder MK, and meet the section GHI in the three points H, I, L; and join the points as in the figure.

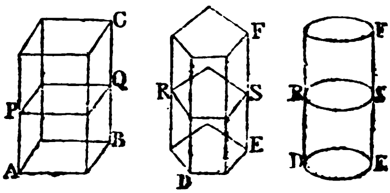
Then, since KL, CI, are parallel (by def. 102); and the plane KI, meeting the two parallel planes ABC, GHI, makes the two sections KC, LI, parallel (th. 105); the figure KLIC is therefore a parallelogram, and consequently has the opposite sides LI, KC, equal, where KC is a radius of the circular base.

In like manner, it is shown that LH is equal to the radius KB; and that any other lines, drawn from the point L to the circumference of the section GHI, are all equal to radii of the base; consequently GHI is a circle, and equal to ABC.

Q. E. D.

THEOREM CVIII.

All Prisms and Cylinders, of Equal Bases and Altitudes, are Equal to each other.



LET AC, DF, be two prisms, and a cylinder, on equal bases AB, DE, and having equal altitudes BC, EF; then will the solids AC, DF, be equal.

For, let PQ, RS, be any

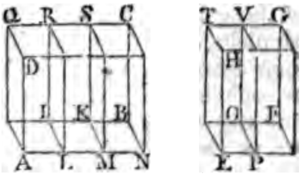
any two sections parallel to the bases, and equidistant from them. Then, by the last two theorems, the section PQ is equal to the base AB, and the section RS equal to the base DE. But the bases AB, DE, are equal, by the hypothesis; therefore the sections PQ, RS, are equal also. In like manner, it may be shown, that any other corresponding sections are equal to one another.

Since then every section in the prism AC, is equal to its corresponding section in the prism or cylinder DF, the prisms and cylinder themselves, which are composed of an equal number or all those equal sections, must also be equal Q.E.D.

Corol. Every prism, or cylinder, is equal to a rectangular parallelopipedon, of an equal base and altitude.

THEOREM CIX.

Rectangular Parallelopipedons, of Equal Altitudes, are to each other as their Bases.



LET AC, EG, be two rectangular parallelopipedons, having the equal altitudes AD, EH; then will the solid AC be to the solid EG, as the base AB is to the base EF.

For, let the proportion of the base AB to the base EF, be that of any one number m (3) to any other number n (2). And conceive AB to be divided into m equal parts, or rectangles, AI, LK, MB (by dividing AN into that number of equal parts, and drawing IL, KM, parallel to BN). And let EF be divided, in like manner, into n equal parts, or rectangles, EO, PF: all of these parts of both bases being mutually equal among themselves. And through the lines of division let the plane sections LR, MS, PV, pass parallel to AQ, ET.

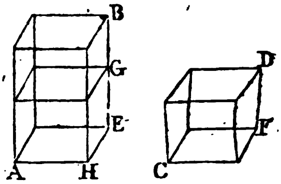
Then, the parallelopipedons AR, LS, MC, EV, PG, are all equal, having equal bases and altitudes. Therefore the solid AC is to the solid EG, as the number of parts in the former, to the number of equal parts in the latter; or as the number of parts in AB to the number of equal parts in EF, that is, as the base AB to the base EF. Q. E. D.

Corol. From this theorem, and the corollary to th[e]
appears, that all prisms and cylinders of equal a[ltitude]

to each other as their bases; every prism and cylinder being equal to a rectangular parallelopipedon of an equal base and altitude.

THEOREM CX.

Rectangular Parallelopipedons, of Equal Bases, are to each other as their Altitudes.



LET AB, CD, be two rectangular parallelopipedons, standing on the equal bases AE, CF; then will the solid AB be to the solid CD, as the altitude EB is to the altitude FD.

For, let AG be a rectangular parallelopipedon on the base AE, and its altitude EG equal to the altitude FD of the solid CD.

Then AG and CD are equal, being prisms of equal bases and altitudes. But if HB, HG, be considered as bases, the solids AB, AG, of equal altitude AH, will be to each other as those bases HB, HG. But these bases HB, HG, being parallelograms of equal altitude HE, are to each other as their bases EB, EG; therefore the two prisms AB, AG, are to each other as the lines EB, EG. But AG is equal to CD, and EG equal to FD; consequently the prisms AC, CD, are to each other as their altitudes EB, FD; that is, - - - AB : CD :: EB : FD. Q. E. D.

Corol. 1. From this theorem, and the corollary to theorem 108, it appears, that all prisms and cylinders, of equal bases, are to one another as their altitudes.

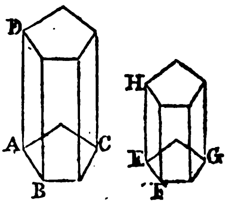
Corol. 2. Because, by corollary 1, prisms and cylinders are as their altitudes, when their bases are equal. And, by the corollary to the last theorem, they are as their bases, when their altitudes are equal. Therefore, universally, when neither are equal, they are to one another as the product of their bases and altitudes. And hence also these products are the proper numeral measures of their quantities or magnitudes.

THEOREM CXI.

Similar Prisms and Cylinders are to each other, as the Cubes of their Altitudes, or of any other Like Linear Dimensions.

LET ABCD, EFGH, be two similar prisms; then will the prism CD be to the prism GH, as AB^3 to EF^3 or AD^3 to EH^3.

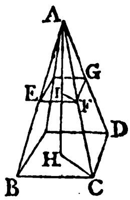
For



For the solids are to each other as the product of their bases and altitudes (th. 110, cor. 2), that is, as AC . AD to EG . EH. But the bases, being similar planes, are to each other as the squares of their like sides, that is, AC to EG as AB^2 to EF^2; therefore the solid CD is to the solid GH, as $AB^2 . AD$ to $FF^2 . EH$. But BD and FH, being similar planes, have their like sides proportional, that is, AB : EF :: AD : EH, - - - - - - or $AB^2 : EF^2 :: AD^2 : EH^2$: therefore $AB^2 . AD : EF^2 . EH :: AB^3 : EF^3$, or :: $AD^3 : EH^3$; conseq. the solid CD : solid GH :: $AB^3 : EF^3 :: AD^3 : EH^3$. Q. E. D.

THEOREM CXII.

In any Pyramid, a Section Parallel to the Base is similar to the Base; and these two planes are to each other as the Squares of their Distances from the Vertex.

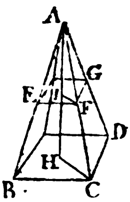


LET ABCD be a pyramid, and EFG a section parallel to the base BCD, also AIH a line perpendicular to the two planes at H and I: then will BD, EG, be two similar planes, and the plane BD will be to the plane EG, as AH^2 to AI^2.

For, join CH, FI. Then, because a plane cutting two parallel planes, makes parallel sections (th. 105), therefore the plane ABC, meeting the two parallel planes BD, EG, makes the sections BC, EF, parallel: In like manner, the plane ACD makes the sections CD, FG, parallel. Again, because two pair of parallel lines make equal angles (th. 104), the two EF, FG, which are parallel to BC, CD, make the angle EFG equal the angle BCD. And in like manner it is shown, that each angle in the plane EG is equal to each angle in the plane BD, and consequently those two planes are equiangular.

Again, the three lines AB, AC, AD, making with the parallels BC, EF, and CD, FG, equal angles (th. 14), and the angles at A being common, the two triangles ABC, AEF, are equiangular, as also the two triangles ACD, AFG, and have therefore their like sides proportional, namely, - - -

AC : AF :: BC : EF :: CD : FG. And in like manner it may be shown, that all the lines in the plane FG, are proportional to all the corresponding lines in the base BD. Hence these two planes, having their angles equal, and their sides proportional, are similar, by def. 68.

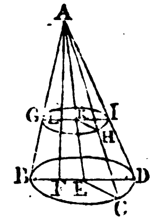


But, similar planes being to each other as the squares of their like sides, the plane BD : EG :: BC^2 : EF^2, or :: AC^2 : AF^2, by what is shown above. Also, the two triangles AHC, AIF, having the angles H and I right ones (th. 98), and the angle A common, are equiangular, and have therefore their like sides proportional, namely, AC : AF :: AH : AI, or AC^2 : AF^2 :: AH^2 : AI^2. Consequently the two planes BD, EG, which are as the former squares AC^2, AF^2, will be also as the latter squares AH^2, AI^2, that is, - - - - - BD : EG :: AH^2 : AI^2. Q. E. D.

THEOREM CXIII.

In a Cone, any Section Parallel to the Base is a Circle; and this Section is to the Base, as the Squares of their Distances from the Vertex.

Let ABCD be a cone, and GHI a section parallel to the base BCD; then will GHI be a circle, and BCD, GHI, will be to each other, as the squares of their distances from the vertex.



For, draw ALF perpendicular to the two parallel planes; and let the planes ACE, ADE, pass through the axis of the cone AKE, meeting the section in the three points H, I, K.

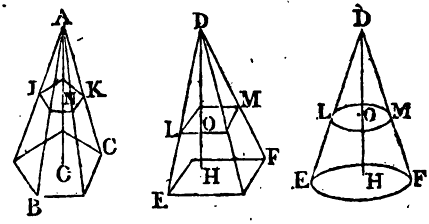
Then, since the section GHI is parallel to the base BCD, and the planes CK, DK, meet them, HK is parallel to CE, and IK to DE (th. 105). And because the triangles formed by these lines are equiangular, KH : EC :: AK : AE :: KI : ED. But EC is equal to ED, being radii of the same circle; therefore KI is also equal to KH. And the same may be shown of any other lines drawn from the point K to the perimeter of the section GHI, which is therefore a circle (def. 44).

Again, by similar triangles, AL : AF :: AK : AE or :: KI : ED, hence AL^2 : AF^2 :: KI^2 : ED^2; but KI^2 : ED^2 :: circle GHI : circle BCD (th. 93); therefore AL^2 : AF^2 :: circle GHI : circle BCD. Q. E. D.

THEOREM

THEOREM CXIV.

All Pyramids, and Cones, of Equal Bases and Altitudes, are Equal to one another.



LET ABC, DEF, be any pyramids and cone, of equal bases BC, EF, and equal altitudes AG, DH: then will the pyramids and cone ABC and DEF, be equal.

For, parallel to the bases and at equal distances AN, DO, from the vertices, suppose the planes IK, LM, to be drawn.

Then, by the two preceding theorems, - - - - - - - -

$DO^2 : DH^2 :: LM : EF$, and

$AN^2 : AG^2 :: IK : BC$.

But since AN^2, AG^2, are equal to DO^2, DH^2,

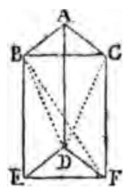
therefore $IK : BC :: LM : EF$. But BC is equal to EF, by hypothesis; therefore IK is also eqnal to LM.

In like manner it is shown, that any other sections, at equal distance from the vertex, are equal to each other.

Since then, every section in the cone, is equal to the corresponding section in the pyramids, and the heights are equal, the solids ABC, DEF, composed of all those sections, must be equal also. Q. E. D.

THEOREM CXV.

Every Pyramid is the Third Part of a Prism of the Same Base and Altitude.

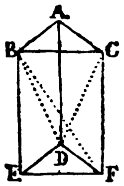


LET ABCDEF be a prism, and BDEF a pyramid, on the same triangular base DEF: then will the pyramid BDEF be a third part of the prism ABCDEF.

For, in the planes of the three sides of the prism, draw the diagonals BF, BD, CD. Then the two planes BDF, BCD, divide the whole prism into the three pyramids BDEF, DABC, DBCF, which are proved to be all equal to one another, as follows.

Since the opposite ends of the prism are equal to each other, the pyramid whose base is ABC and vertex D, is equal to the pyramid

pyramid whose base is DEF and vertex B (th. 114), being pyramids of equal base and altitude.



But the latter pyramid, whose base is DEF and vertex B, is the same solid as the pyramid whose base is BEF and vertex D, and this is equal to the third pyramid whose base is BCF and vertex D, being pyramids of the same altitude and equal bases BEF, BCF.

Consequently all the three pyramids, which compose the prism, are equal to each other, and each pyramid is the third part of the prism, or the prism is triple of the pyramid. Q. E. D.

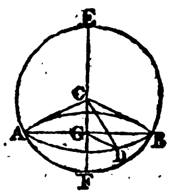
Hence also, every pyramid, whatever its figure may be, is the third part of a prism of the same base and altitude; since the base of the prism, whatever be its figure, may be divided into triangles, and the whole solid into triangular prisms and pyramids.

Corol. Any cone is the third part of a cylinder, or of a prism, of equal base and altitude; since it has been proved that a cylinder is equal to a prism, and a cone equal to a pyramid, of equal base and altitude.

Scholium. Whatever has been demonstrated of the proportionality of prisms, or cylinders, holds equally true of pyramids, or cones; the former being always triple the latter; viz. that similar pyramids or cones are as the cubes of their like linear sides, or diameters, or altitudes, &c. And the same for all similar solids whatever, viz. that they are in proportion to each other, as the cubes of their like linear dimensions, since they are composed of pyramids every way similar.

THEOREM CXVI.

If a Sphere be cut by a Plane, the Section will be a Circle.



LET the sphere AEBF be cut by the plane ADB; then will the section ADB be a circle.

Draw the chord AB, or diameter of the section; perpendicular to which, or to the section ADB, draw the axis of the sphere ECGF, through the centre C, which will bisect the chord AB in the point G (th. 41). Also, join CA, CB;

and

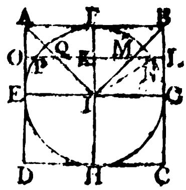
and draw CD, GD, to any point D in the perimeter of the section ADB.

Then, because CG is perpendicular to the plane ADB, it is perpendicular both to GA and GD (def. 90). So that CGA, CGD are two right-angled triangles, having the perpendicular CG common, and the two hypothenuses CA, CD, equal, being both radii of the sphere; therefore the third sides GA, GD, are also equal (cor. 2, th. 34). In like manner it is shown, that any other line, drawn from the centre G to the circumference of the section ADB, is equal to GA or GB; consequently that section is a circle.

Corol. The section through the centre, is a circle having the same centre and diameter as the sphere, and is called a great circle of the sphere; the other plane sections being little circles.

THEOREM CXVII.

Every Sphere is Two-Thirds of its Circumscribing Cylinder.

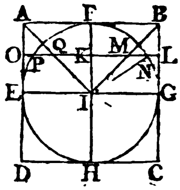


LET ABCD be a cylinder, circumscribing the sphere EFGH; then will the sphere EFGH be two-thirds of the cylinder ABCD.

For, let the plane AC be a section of the sphere and cylinder through the centre I. Join AI, BI. Also, let FIH be parallel to AD or BC, and EIG and KL parallel to AB or DC, the base of the cylinder; the latter line KL meeting BI in M, and the circular section of the sphere in N.

Then, if the whole plane HFBC be conceived to revolve about the line HF as an axis, the square FG will describe a cylinder AG, and the quadrant IFG will describe a hemisphere EFG, and the triangle IFB will describe a cone IAB. Also, in the rotation, the three lines or parts KL, KN, KM, as radii, will describe corresponding circular sections of those solids, namely, KL a section of the cylinder, KN a section of the sphere, and KM a section of the cone.

Now, FB being equal to FI or IG, and KL parallel to FB, then by similar triangles IK is equal to KM (th. 82). And since, in the right-angled triangle IKN, IN^2 is equal to $IK^2 + KN^2$ (th. 34); and because KL is equal to the radius IG or



or IN, and KM = IK, therefore KL^2 is equal to $KM^2 + KN^2$, or the square of the longest radius, of the said circular sections, is equal to the sum of the squares of the two others. And because circles are to each other as the squares of their diameters, or of their radii, therefore the circle described by KL is equal to both the circles described by KM and KN; or the section of the cylinder, is equal to both the corresponding sections of the sphere and cone. And as this is always the case in every parallel position of KL, it follows, that the cylinder EB, which is composed of all the former sections, is equal to the hemisphere EFG and cone IAB, which are composed of all the latter sections.

But the cone IAB is a third part of the cylinder EB (cor. 2, th. 115); consequently the hemisphere EFG is equal to the remaining two-thirds; or the whole sphere EFGH equal to two-thirds of the whole cylinder ABCD. Q. E. D.

Corol. 1. A cone, hemisphere, and cylinder of the same base and altitude, are to each other as the numbers 1, 2, 3.

Corol. 2. All spheres are to each other as the cubes of their diameters; all these being like parts of their circumscribing cylinders.

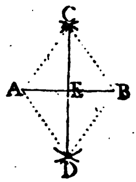
Corol. 3. From the foregoing demonstration it also appears, that the spherical zone or frustrum EGNP, is equal to the difference between the cylinder EGLO and the cone IMQ, all of the same common height IK. And that the spherical segment PFN, is equal to the difference between the cylinder ABLO and the conic frustrum AQMB, all of the same common altitude FK.

PROBLEMS.

PROBLEMS.

PROBLEM I.

To Bisect a Line AB; that is, to divide it into two Equal Parts.



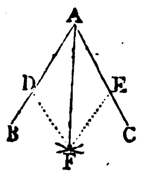
FROM the two centres A and B, with any equal radii, describe arcs of circles, intersecting each other in C and D; and draw the line CD, which will bisect the given line AB in the point E.

For, draw the radii AC, BC, AD, BD. Then, because all these four radii are equal, and the side CD common, the two triangles ACD, BCD, are mutually equilateral: consequently they are also mutually equiangular (th. 5), and have the angle ACE equal to the angle BCE.

Hence, the two triangles ACE, BCE, having the two sides AC, CE, equal to the two sides BC, CE, and their contained angles equal, are identical (th. 1), and therefore have the side AE equal to EB. Q. E. D.

PROBLEM II.

To Bisect an Angle BAC.



FROM the centre A, with any radius, describe an arc, cutting off the equal lines AD, AE; and from the two centres D, E, with the same radius, describe arcs intersecting in F; then draw AF, which will bisect the angle A as required.

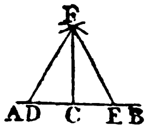
For, join DF, EF. Then the two triangles ADF, AEF, having the two sides AD, DF, equal to the two AE, EF (being equal radii), and the side AF common, they are mutually equilateral; consequently they are also mutually equiangular (th. 5), and have the angle BAF equal to the angle CAF.

Scholium. In the same manner is an arc of a circle bi-

PROBLEM

PROBLEM III.

At a Given Point C, in a Line AB, to Erect a Perpendicular.

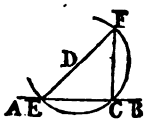


FROM the given point C, with any radius, cut off any equal parts CD, CE, of the given line; and, from the two centres D and E, with any one radius, describe arcs intersecting in F; then join CF, which will be perpendicular as required.

For, draw the two equal radii DF, EF. Then the two triangles CDF, CEF, having the two sides CD, DF, equal to the two CE, EF, and CF common, are mutually equilateral; consequently they are also mutually equiangular (th. 5), and have the two adjacent angles at C equal to each other; therefore the line CF is perpendicular to AB (def. 11).

Otherwise.

When the Given Point C is near the End of the Line.

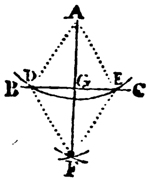


FROM any point D, assumed above the line, as a centre, through the given point C describe a circle, cutting the given line at E; and through E and the centre D, draw the diameter EDF; then join CF, which will be the perpendicular required.

For the angle at C, being an angle in a semicircle, is a right angle, and therefore the line CF is a perpendicular (by def. 15).

PROBLEM IV.

From a Given Point A, to let fall a Perpendicular on a given Line BC.



FROM the given point A as a centre, with any convenient radius, describe an arc, cutting the giving line at the two points D and E; and from the two centres D, E, with any radius, describe two arcs, intersecting at F; then draw AGF, which will be perpendicular to BC as required.

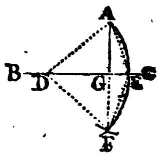
For, draw the equal radii AD, AE, and DF, EF. Then the two triangles ADF, AEF, having the two sides AD, DF, equal to the two AE, EF, and AF common, are

mutually

mutually equilateral; consequently they are also mutually equiangular (th. 5), and have the angle DAG equal the angle EAG. Hence then, the two triangles ADG, AEG, having the two sides AD, AG, equal to the two AE, AG, and their included angles equal, are therefore equiangular (th. 1), and have the angles at G equal; consequently AG is perpendicular to BC (def. 11).

Otherwise.

When the Given Point is nearly Opposite the end of the Line.

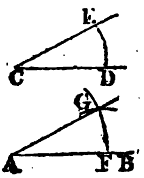


FROM any point D, in the given line BC, as a centre, describe the arc of a circle through the given point A, cutting BC in E; and from the centre E, with the radius EA, describe another arc, cutting the former in F; then draw AGF, which will be perpendicular to BC as required.

For, draw the equal radii DA, DF, and EA, EF. Then the two triangles DAE, DFE, will be mutually equilateral; consequently they are also mutually equiangular (th. 5), and have the angles at D equal. Hence, the two triangles DAG, DFG, having the two sides DA, DG, equal to the two DF, DG, and the included angles at D equal, have also the angles at G equal (th. 1); consequently those angles at G are right angles, and the line AG is perpendicular to DG.

PROBLEM V.

At a Given Point A, in a Line AB, to make an Angle Equal to a Given Angle C.



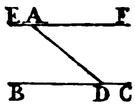
FROM the centres A and C, with any one radius, describe the arcs DE, FG. Then, with radius DE, and centre F, describe an arc, cutting FG in G. Through G draw the line AG, and it will form the angle required.

For, conceive the equal lines or radii, DE, FG, to be drawn. Then the two triangles CDE, AFG, being mutually equilateral, are mutually equiangular (th. 5), and have the angle at A equal to the angle C.

PROBLEM

PROBLEM VI.

Through a Given Point A, to draw a Line Parallel to a Given Line BC.

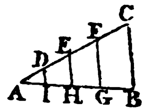


From the given point A draw a line AD to any point in the given line BC. Then draw the line EAF making the angle at A equal to the angle at D (by prob. 5); so shall EF be parallel to BC as required.

For, the angle D being equal to the alternate angle A, the lines BC, EF, are parallel, by th. 13.

PROBLEM VII.

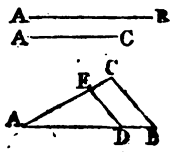
To Divide a Line AB into any proposed Number of Equal Parts.



Draw any other line AC, forming any angle with the given line AB; on which set off as many of any equal parts, AD, DE, EF, FC, as the line AB is to be divided into. Join BC; parallel to which draw the other lines FG, EH, DI: then these will divide AB in the manner as required.—For those parallel lines divide both the sides AB, AC, proportionally, by th. 82.

PROBLEM VIII.

To find a Third Proportional to Two given Lines AB, AC.



Place the two given lines AB, AC, forming any angle at A; and in AB take also AD equal to AC. Join BC, and draw DE parallel to it; so will AE be the third proportional sought.

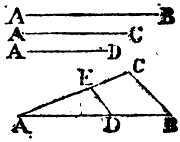
For, because of the parallels BC, DE, the two lines AB, AC, are cut proportionally (th. 82); so that AB : AC :: AD or AC : AE; therefore AE is the third proportional to AB, AC.

PROBLEM IX.

To find a Fourth Proportional to three Lines AB, AC, AD.

Place two of the given lines AB, AC, making any ? at A; also place AD on AB. Join BC; and parallel to

to it draw DE: so shall AE be the fourth proportional as required.

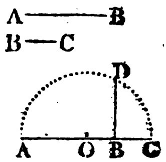


For, because of the parallels BC, DE, the two sides AB, AC, are cut proportionally (th. 82); so that - - - - AB : AC : : AD : AE.

PROBLEM X.

To find a Mean Proportional between Two Lines AB, BC.

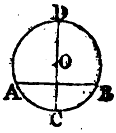
PLACE AB, BC, joined in one straight line AC: on which, as a diameter, describe the semicircle ADC; to meet which erect the perpendicular BD; and it will be the mean proportional sought, between AB and BC (by cor. th. 87).



PROBLEM XI.

To find the Centre of a Circle.

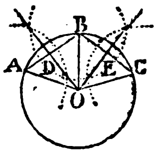
DRAW any chord AB; and bisect it perpendicularly with the line CD, which will be a diameter (th. 41, cor.). Therefore CD bisected in O, will give the centre, as required.



PROBLEM XII.

To describe the Circumference of a Circle through Three Given Points A, B, C.

FROM the middle point B draw chords BA, BC, to the two other points, and bisect these chords perpendicularly by lines meeting in O, which will be the centre. Then from the centre O, at the distance of any one of the points, as OA, describe a circle, and it will pass through the two other points B, C, as required.

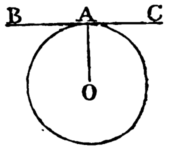


For, the two right-angled triangles OAD, OBD, having the sides AD, DB, equal (by constr.), and OD common with the included right angles at D equal, have their third sides OA, OB, also equal (th. 1). And in like manner it is shown, that OC is equal to OB or OA. So that all the three OA, OB, OC, being equal, will be radii of the same circle.

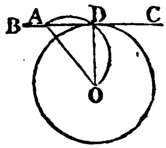
PROBLEM

PROBLEM XIII.

To draw a Tangent to a Circle, through a Given Point A.



When the given point A is in the circumference of the circle: Join A and the centre O; perpendicular to which draw BAC, and it will be the tangent, by th. 46.

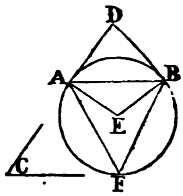


But when the given point A is out of the circle: Draw AO to the centre O; on which as a diameter describe a semicircle, cutting the given circumference in D; through which draw BADC, which will be the tangent as required.

For, join DO. Then the angle ADO, in a semicircle, is a right angle, and consequently AD is perpendicular to the radius DO, or is a tangent to the circle (th. 46).

PROBLEM XIV.

On a Given Line B to describe a Segment of a Circle, to Contain a Given Angle C.

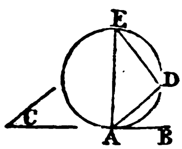


At the ends of the given line make angles DAB, DBA, each equal to the given angle C. Then draw AE, BE, perpendicular to AD, BD; and with the centre E, and radius EA or EB, describe a circle; so shall AFB be the segment required, as any angle F made in it will be equal to the given angle C.

For, the two lines AD, BD, being perpendicular to the radii EA, EB (by constr.), are tangents to the circle (th. 46); and the angle A or B, which is equal to the given angle C by construction, is equal to the angle F in the alternate segment AFB (th. 53).

PROBLEM XV.

To Cut off a Segment from a Circle, that shall Contain a Given Angle C.



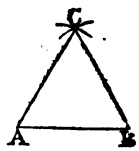
Draw any tangent AB to the given circle; and a chord AD to make the angle DAB equal to the given angle C; then DEA will be the segment required, any angle E made in it being equal to given angle C.

For

For the angle A, made by the tangent and chord, which is equal to the given angle C by construction, is also equal to any angle E in the alternate segment (th. 53).

PROBLEM XVI.

To make an Equilateral Triangle on a Given Line AB.

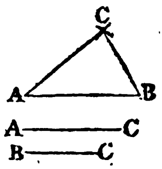


FROM the centres A and B, with the distance AB, describe arcs, intersecting in C. Draw AC, BC, and ABC will be the equilateral triangle.

For the equal radii AC, BC, are, each of them, equal to AB.

PROBLEM XVII.

To make a Triangle with Three Given Lines AB, AC, BC.

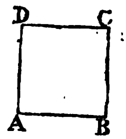


WITH the centre A, and distance AC, describe an arc. With the centre B, and distance BC, describe another arc, cutting the former in C. Draw AC, BC, and ABC will be the triangle required.

For the radii, or sides of the triangle, AC, BC, are equal to the given lines AC, BC, by construction.

PROBLEM XVIII.

To make a Square on a Given Line AB.

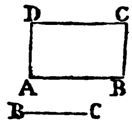


RAISE AD, BC, each perpendicular and equal to AB; and join DC; so shall ABCD be the square sought.

For all the three sides AB, AD, BC, are equal, by the construction, and DC is equal and parallel to AB (by th. 24); so that all the four sides are equal, and the opposite ones are parallel. Again, the angle A or B, of the parallelogram, being a right angle, the angles are all right ones (cor. 1, th. 22). Hence, then, the figure, having all its sides equal, and all its angles right, is a square (def. 34).

PROBLEM XIX.

To make a Rectangle, or a Parallelogram, of a Given Length and Breadth, AB, BC.



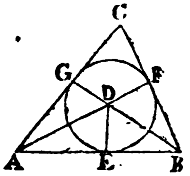
ERECT AD, BC, perpendicular to AB, and each equal to BC; then join DC, and it is done.

The demonstration is the same as the last problem.

And in the same manner is described any oblique parallelogram, only drawing AD and BC to make the given oblique angle with AB, instead of perpendicular to it.

PROBLEM XX.

To Inscribe a Circle in a Given Triangle ABC.



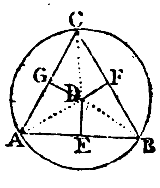
BISECT any two angles A and B, with the two lines AD, BD. From the intersection D, which will be the centre of the circle, draw the perpendiculars DE, DF, DG, and they will be the radii of the circle required.

For, since the angle DAE is equal to the angle DAG, and the angles at E, G, right angles (by constr.), the two triangles ADE, ADG, are equiangular; and, having also the side AD common, they are identical, and have the sides DE, DG, equal (th. 2). In like manner it is shown, that DF is equal to DE or DG.

Therefore, if with the centre D, and distance DE, a circle be described, it will pass through all the three points E, F, G, in which points also it will touch the three sides of the triangle (th. 46), because the radii DE, DF, DG, are perpendicular to them.

PROBLEM XXI.

To Describe a Circle about a Given Triangle ABC.



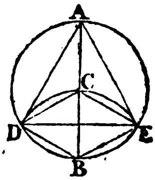
BISECT any two sides with two of the perpendiculars DE, DF, DG, and D will be the centre.

For, join DA, DB, DC. Then the two right-angled triangles DAE, DBE, have the two sides DE, EA, equal to the two DE, EB, and the included angles at E equal: those two triangles are therefore identical

(th. 1), and have the side DA equal to DB. In like manner it is shown, that DC is also equal to DA or DB. So that all the three DA, DB, DC, being equal, they are radii of a circle passing through A, B, and C.

PROBLEM XXII.

To Inscribe an Equilateral Triangle in a Given Circle.

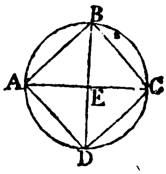


THROUGH the centre C draw any diameter AB. From the point B as a centre, with the radius BC of the given circle, describe an arc DCE. Join AD, AE, DE, and ADE is the equilateral triangle sought.

For, join DB, DC, EB, EC. Then DCB is an equilateral triangle, having each side equal to the radius of the given circle. In like manner, BCE is an equilateral triangle. But the angle ADE is equal to the angle ABE or CBE, standing on the same arc AE; also the angle AED is equal to the angle CBD, on the same arc AD; hence the triangle DAE has two of its angles, ADE, AED, equal to the angles of an equilateral triangle, and therefore the third angle at A is also equal to the same; so that triangle is equiangular, and therefore equilateral.

PROBLEM XXIII.

To Inscribe a Square in a Given Circle.



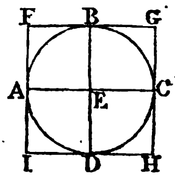
DRAW two diameters AC, BD, crossing at right angles in the centre E. Then join the four extremities A, B, C, D, with right lines, and these will form the inscribed square ABCD.

For the four right-angled triangles AEB, BEC, CED, DEA, are identical, because they have the sides EA, EB, EC, ED, all equal, being radii of the circle, and the four included angles at E all equal, being right angles, by the construction. Therefore all their third sides AB, BC, CD, DA, are equal to one another, and the figure ABCD is equilateral. Also, all its four angles, A, B, C, D, are right ones, being angles in a semicircle. Consequently the figure is a square.

PROB-

PROBLEM XXIV.

To Describe a Square about a Given Circle.



DRAW two diameters AC, BD, crossing at right angles in the centre E. Then through their four extremities draw FG, IH, parallel to AC, and FI, GH, parallel to BD, and they will form the square FGHI.

For, the opposite sides of parallelograms being equal, FG and IH are each equal to the diameter AC, and FI and GH each equal to the diameter BD; so that the figure is equilateral. Again, because the opposite angles of parallelograms are equal, all the four angles F, G, H, I, are right angles, being equal to the opposite angles at E. So that the figure FGHI, having its sides equal, and its angles right ones, is a square, and its sides touch the circle at the four points A, B, C, D, being perpendicular to the radii drawn to those points.

PROBLEM XXV.

To Inscribe a Circle in a Given Square.

BISECT the two sides FG, FI, in the points A and B (last fig.). Then through these two points draw AC parallel to FG or IH, and BD parallel to FI or GH. Then the point of intersection E will be the centre, and the four lines EA, EB, EC, ED, radii of the inscribed circle.

For, because the four parallelograms EF, EG, EH, EI, have their opposite sides and angles equal, therefore all the four lines EA, EB, EC, ED, are equal, being each equal to half a side of the square. So that a circle described from the centre E, with the distance EA, will pass through all the points A, B, C, D, and will be inscribed in the square, or will touch its four sides in those points, because the angles there are right ones.

PROBLEM XXVI.

To Describe a Circle about a Given Square, (see fig. Prob. xxiii).

DRAW the diagonals AC, BD, and their intersection E will be the centre.

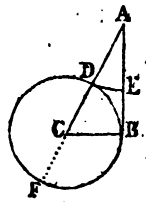
For the diagonals of a square bisect each other (th. 40), making EA, EB, EC, ED, all equal, and consequently these are radii of a circle passing through the four points A, B, C, D.

PROBLEM

PROBLEM XXVII.

To Cut a Given Line in Extreme and Mean Ratio.

LET AB be the given line to be divided in extreme and mean ratio, that is, so as that the whole line may be to the greater part, as the greater part is to the less part.



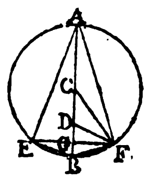
Draw BC perpendicular to AB, and equal to half AB. Join AC; and with centre C and distance CB, describe the circle BD; then with centre A and distance AD, describe the arc DE; so shall AB be divided in E in extreme and mean ratio, or so that AB : AE :: AE : EB.

For, produce AC to the circumference at F. Then, ADF being a secant, and AB a tangent, because B is a right angle: therefore the rectangle AF.AD is equal to AB^2 (cor. 1 th. 61); consequently the means and extremes of these are proportional (th. 77), viz. AB : AF or AD + DF : : AD : AB. But AE is equal to AD by construction, and AB = 2BC = DF; therefore, AB : AE + AB : : AE : AB; and by division, AB : AE : : AE : EB.

PROBLEM XXVIII.

To Inscribe an Isosceles Triangle in a Given Circle, that shall have each of the Angles at the Base Double the Angle at the Vertex.

DRAW any diameter AB of the given circle; and divide the radius CB, in the point D, in extreme and mean ratio, by the last problem. From the point B apply the chords BE, BF, each equal to the greater part CD. Then join AE, AF, EF; and AEF will be the triangle required.



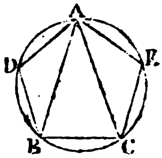
For, the chords BE, BF, being equal, their arcs are equal; therefore the supplemental arcs and chords AE, AF, are also equal; consequently the triangle AEF is isosceles, and has the angle E equal to the angle F; also the angles at G are right angles.

Draw CF and DF. Then, BC : CD : : CD : BD, or BC : BF : : BF : BD by constr. And BA : BF : : BF : BG (by th. 87). But BC = $\frac{1}{2}$BA; therefore BG = $\frac{1}{2}$BD = GD; therefore the two triangles GBF, GDF, are identical (th. 1), and

and each equiangular to ABF and AGF (th. 87). Therefore their doubles, BFD, AFE, are isosceles and equiangular, as well as the triangle BCF; having the two sides BC, CF, equal, and the angle B common with the triangle BFD. But CD is = DF or BF; therefore the angle C = the angle DFC (th. 4); consequently the angle BDF, which is equal to the sum of these two equal angles (th. 16), is double of one of them C; or the equal angle B or CFB double the angle C. So that CBF is an isosceles triangle, having each of its two equal angles double of the third angle C. Consequently the triangle AEF (which it has been shown is equiangular to the triangle CBF) has also each of its angles at the base double the angle A at the vertex.

PROBLEM XXIX.

To Inscribe a Regular Pentagon in a Given Circle.



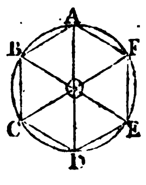
INSCRIBE the isosceles triangle ABC having each of the angles ABC, ACB, double the angle BAC (prob. 28). Then bisect the two arcs ADB, AEC, in the points D, E; and draw the chords AD, DB, AE, EC, so shall ADBCE be the inscribed equilateral pentagon required.

For, because equal angles stand on equal arcs, and double angles on double arcs, also the angles ABC, ACB, being each double the angle BAC, therefore the arcs ADB, AEC, subtending the two former angles, one each double the arcs BC subtending the latter. And since the two former arcs are bisected in D and E, it follows that all the five arcs AD, DB, BC, CE, EA, are equal to each other, and consequently the chords also which subtend them, or the five sides of the pentagon, are all equal.

Note. In the construction, the points D and E are most easily found, by applying BD and CE each equal to BC.

PROBLEM XXX.

To Inscribe a Regular Hexagon in a Circle.



APPLY the radius AO of the given circle as a chord, AB, BC, CD, &c, quite round the circumference, and it will complete the regular hexagon ABCDEF.

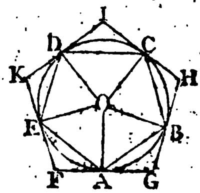
For, draw the radii AO, BO, CO, DO, EO, FO, completing six equal triangles; of which any one, as ABO, being equilateral

(By constr.) its three angles are all equal (cor. 2, th. 3), and any one of them, as AOB, is one-third of the whole, or of two right angles (th. 17), or one-sixth of four right angles. But the whole circumference is the measure of four right angles (cor. 4, th. 6). Therefore the arc AB is one-sixth of the circumference of the circle, and consequently its chord AB one side of an equilateral hexagon inscribed in the circle. And the same of the other chords.

Corol. The side of a regular hexagon is equal to the radius of the circumscribing circle, or to the chord of one-sixth part of the circumference.

PROBLEM XXXI.

To describe a Regular Pentagon or Hexagon about a Circle.



In the given circle inscribe a regular polygon of the same name or number of sides, as ABCDE, by one of the foregoing problems. Then to all its angular points draw tangents (by prob. 13), and these will form the circumscribing polygon required.

For, all the chords, or sides of the inscribing figure, AB, BC, &c, being equal, and all the radii OA, OB, &c, being equal, all the vertical angles about the point O are equal. But the angles OEF, OAF, OAG, OBG, made by the tangents and radii, are right angles; therefore OEF + OAF = two right angles, and OAG + OBG = two right angles; consequently, also, AOE + AFE = two right angles, and AOB + AGB = two right angles (cor. 2, th. 18). Hence, then, the angles AOE + AFE being = AOB + AGB, of which AOB is = AOE; consequently the remaining angles F and G are also equal. In the same manner it is shown, that all the angles F, G, H, I, K, are equal.

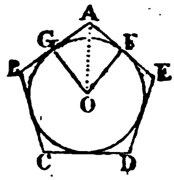
Again, the tangents from the same point FE, FA, are equal, as also the tangents AG, GB (cor. 2, th. 61,); and the angles F and G of the isosceles triangles AFE, AGB, are equal, as well as their opposite sides AE, AB; consequently those two triangles are identical (th. 1), and have their other sides EF, FA, AG, GB, all equal, and FG equal to the double of any one of them. In like manner it is shown, that all the other sides GH, HI, IK, KF, are equal to FG, or double of the tangents GB, BH, &c.

Hence, then, the circumscribed figure is both equilateral and equiangular, which was to be shown.

Corol. The inscribed circle touches the middles of the sides of the polygon.

PROBLEM XXXII.

To Inscribe a Circle in a Regular Polygon.

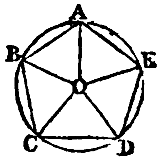


Bisect any two sides of the polygon by the perpendiculars GO, FO, and their intersection O will be the centre of the inscribed circle, and OG or OF will be the radius.

For the perpendiculars to the tangents AF, AG, pass through the centre (cor. th. 47); and the inscribed circle touches the middle points F, G, by the last corollary. Also, the two sides AG, AO, of the right-angled triangle AOG, being equal to the two sides AF, AO, of the right-angled triangle AOF, the third sides OF, OG, will also be equal (cor. th. 45). Therefore the circle described with the centre O and radius OG, will pass through F, and will touch the sides in the points G and F. And the same for all the other sides of the figure.

PROBLEM XXXIII.

To Describe a Circle about a Regular Polygon.



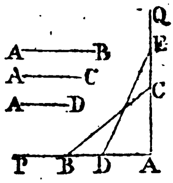
Bisect any two of the angles, C and D, with the lines CO, DO; then their intersection O will be the centre of the circumscribing circle; and OC, or OD, will be the radius.

For, draw OB, OA, OE, &c, to the angular points of the given polygon. Then the triangle OCD is isosceles, having the angles at C and D equal, being the halves of the equal angles of the polygon BCD, CDE; therefore their opposite sides CO, DO, are equal (th. 4). But the two triangles OCD, OCB, having the two sides OC, CD, equal to the two OC, CB, and the included angles OCD, OCB, also equal, will be identical (th. 1), and have their third sides BO, OD, equal. In like manner it is shown, that all the lines OA, OB, OC, OD, OE, are equal. Consequently a circle described with the centre O and radius OA, will pass through all the other angular points, B, C, D, &c, and will circumscribe the polygon.

PROBLEM

PROBLEM XXXIV.

To make a Square Equal to the Sum of two or more Given Squares.

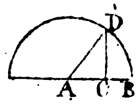


Let AB and AC be the sides of two given squares. Draw two indefinite lines AP, AQ, at right angles to each other; in which place the sides AB, AC, of the given squares; join BC; then a square described on BC will be equal to the sum of the two squares described on AB and AC (th. 34).

In the same manner, a square may be made equal to the sum of the three or more given squares. For, if AB, AC, AD, be taken as the sides of the given squares, then, making AE = BC, AD = AD, and drawing DE, it is evident that the square on DE will be equal to the sum of the three squares on AB, AC, AD. And so on for more squares.

PROBLEM XXXV.

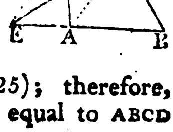
To make a Square Equal to the Difference of two Given Squares.



Let AB and AC, taken in the same straight line, be equal to the sides of the two given squares.—From the centre A, with the distance AB, describe a circle; and make CD perpendicular to AB, meeting the circumference in D: so shall a square described on CD be equal to $AD^2 - AC^2$, or $AB^2 - AC^2$, as required (cor. th. 34).

PROBLEM XXXVI.

To make a Triangle Equal to a Given Quadrangle ABCD.



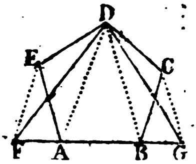
Draw the diagonal AC, and parallel to it DE, meeting BA produced at E, and join CE; then will the triangle CEB be equal to the given quadrilateral ABCD.

For, the two triangles ACE, ACD, being on the same base AC, and between the same parallels AC, DE, are equal (th. 25); therefore, if ABC be added to each, it will make BCE equal to ABCD (ax. 2).

PROBLEM

PROBLEM XXXVII.

To make a Triangle Equal to a Given Pentagon ABCDE.

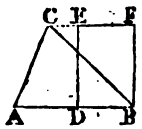


DRAW DA and DB, and also EF, CG, parallel to them, meeting AB produced at F and G; then draw DF and DG; so shall the triangle DFG be equal to the given pentagon ABCDE.

For the triangle DFA = DEA, and the triangle DGB = DCB (th. 25); therefore, by adding DAB to the equals, the sums are equal (ax. 2), that is, DAB + DAF + DBG = DAB + DAE + DBC, or the triangle DFG = to the pentagon ABCDE.

PROBLEM XXXVIII.

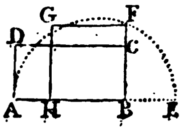
To make a Rectangle Equal to a Given Triangle ABC.



BISECT the base AB in D; then raise DE and BF perpendicular to AB, and meeting CF parallel to AB, at E and F: so shall DF be the rectangle equal to the given triangle ABC (by cor. 2, th. 26).

PROBLEM XXXIX.

To make a Square Equal to a Given Rectangle ABCD.



PRODUCE one side AB, till BE be equal to the other side BC. On AE as a diameter describe a circle, meeting BC produced at F: then will BF be the side of the square BFGH, equal to the given rectangle BD, as required; as appears by cor. th. 87, and th. 77.

APPLICATION

APPLICATION OF ALGEBRA

TO

GEOMETRY.

WHEN it is proposed to resolve a geometrical problem algebraically, or by algebra, it is proper, in the first place, to draw a figure that shall represent the several parts or conditions of the problem, and to suppose that figure to be the true one. Then, having considered attentively the nature of the problem, the figure is next to be prepared for a solution, if necessary, by producing or drawing such lines in it as appear most conducive to that end. This done, the usual symbols or letters, for known and unknown quantities, are employed to denote the several parts of the figure, both the known and unknown parts, or as many of them as necessary, as also such unknown line or lines as may be easiest found, whether required or not. Then proceed to the operation, by observing the relations that the several parts of the figure have to each other; from which, and the proper theorems in the foregoing elements of geometry, make out as many equations independent of each other, as there are unknown quantities employed in them: the resolution of which equations, in the same manner as in arithmetical problems, will determine the unknown quantities, and resolve the problem proposed.

As no general rule can be given for drawing the lines, and selecting the fittest quantities to substitute for, so as always to bring out the most simple conclusions, because different problems require different modes of solution; the best way to gain experience, is to try the solution of the same problem in different ways, and then apply that which succeeds best, to other cases of the same kind, when they afterwards occur. The following particular directions, however, may be of some use.

1*st*, In preparing the figure, by drawing lines, let them be either parallel or perpendicular to other lines in the figure, or so as to form similar triangles. And if an angle be given, it will be proper to let the perpendicular be opposite to that angle, and to fall from one end of a given line, if possible.

2d,

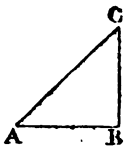
2*d*, In selecting the quantities proper to substitute for, those are to be chosen, whether required or not, which lie nearest the known or given parts of the figure, and by means of which the next adjacent parts may be expressed by addition and subtraction only, without using surds.

3*d*, When two lines or quantities are alike related to other parts of the figure or problem, the best way is, not to make use of either of them separately, but to substitute for their sum, or difference, or rectangle, or the sum of their alternate quotients, or for some line or lines, in the figure, to which they have both the same relation.

4*th*, When the area, or the perimeter, of a figure, is given, or such parts of it as have only a remote relation to the parts required: it is sometimes of use to assume another figure similar to the proposed one, having one side equal to unity, or some other known quantity. For, hence the other parts of the figure may be found, by the known proportions of the like sides, or parts, and so an equation be obtained. For examples, take the following problems.

PROBLEM I.

In a Right-angled Triangle, having given the Base (3), *and the Sum of the Hypothenuse and Perpendicular* (9); *to find both these two Sides.*



LET ABC represent the proposed triangle, right-angled at B. Put the base AB $= 3 = b$, and the sum AC + BC of the hypothenuse and perpendicular $= 9 = s$; also, let x denote the hypothenuse AC, and y the perpendicular BC.

Then by the question - - - $x + y = s$,
and by theorem 34, - - - - $x^2 = y^2 + b^2$.
By transpos. y in the 1st equ. gives $x = s - y$,
This value of x substi. in the 2d,
gives - - - - - $s^2 - 2sy + y^2 = y^2 + b^2$,
Taking away y^2 on both sides leaves $s^2 - 2sy = b^2$,
By transpos. $2sy$ and b^2, gives - $s^2 - b^2 = 2sy$,
And dividing by $2s$, gives - - $\frac{s^2 - b^2}{2s} = y = 4 =$ BC.
Hence $x = s - y = 5 =$ AC.

N. B. In this solution, and the following ones, the notation is made by using as many unknown letters, x and y, as there

there are unknown sides of the triangle, a separate letter for each; in preference to using only one unknown letter for one side, and expressing the other unknown side in terms of that letter and the given sum or difference of the sides; though this latter way would render the solution shorter and sooner; because the former way gives occasion for more and better practice in reducing equations, which is the very end and reason for which these problems are given at all.

PROBLEM II.

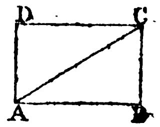
In a Right-angled Triangle, having given the Hypothenuse (5); and the Sum of the Base and Perpendicular (7); to find both these two Sides.

LET ABC represent the proposed triangle, right-angled at B. Put the given hypothenuse AC $= 5 = a$, and the sum AB + BC of the base and perpendicular $= 7 = s$; also let x denote the base AB, and y the perpendicular BC.

Then by the question - - - $x + y = s$
and by theorem 34 - - - - $x^2 + y^2 = a^2$
By transpos. y in the 1st, gives $x = s - y$
By substitu. this valu. for x, gives $s^2 - 2sy + 2y^2 = a^2$
By transposing s^2, gives - - $2y^2 - 2sy = a^2 - s^2$
By dividing by 2, gives - - $y^2 - sy = \frac{1}{2}a^2 - \frac{1}{2}s^2$
By completing the square, gives $y^2 - sy + \frac{1}{4}s^2 = \frac{1}{2}a^2 - \frac{1}{4}s^2$
By extracting the root, gives - $y - \frac{1}{2}s = \sqrt{\frac{1}{2}a^2 - \frac{1}{4}s^2}$
By transposing $\frac{1}{2}s$, gives - - $y = \frac{1}{2}s \pm \sqrt{\frac{1}{2}a^2 - \frac{1}{4}s^2} =$
4 and 3, the values of x and y.

PROBLEM III.

In a Rectangle, having given the Diagonal (10), and the Perimeter, or Sum of all the Four Sides (28); to find each of the Sides severally.



LET ABCD be the proposed rectangle; and put the diagonal AC $= 10 = d$, and half the perimeter AB + BC or AD + DC $= 14 = a$; also put one side AB $= x$, and the other side BC $= y$. Hence, by right-angled triangles, - - - - $x^2 + y^2 = d^2$
And by the question - - - - $x + y = a$
Then by transposing y in the 2d, gives $x = a - y$
This value substituted in the 1st, gives $a^2 - 2ay + 2y^2 = d^2$

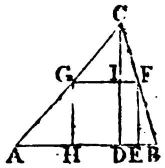
Transposing

Transposing a^2, gives - - - $2y^2 - 2ay = d^2 - a^2$
And dividing by 2, gives - - $y^2 - ay = \frac{1}{2}d^2 - \frac{1}{2}a^2$
By completing the square, it is $y^2 - ay + \frac{1}{4}a^2 = \frac{1}{2}d^2 - \frac{1}{4}a^2$
And extracting the root, gives $y - \frac{1}{2}a = \sqrt{\frac{1}{2}d^2 - \frac{1}{4}a^2}$
And transposing $\frac{1}{2}a$, gives - $y = \frac{1}{2}a \pm \sqrt{\frac{1}{2}d^2 - \frac{1}{4}a^2} = 8$ or 6, the values of x and y.

PROBLEM IV.

Having given the Base and Perpendicular of any Triangle; to find the Side of a Square Inscribed in the same.

LET ABC represent the given triangle, and EFGH its inscribed square. Put the base AB $= b$, the perpendicular CD $= a$, and the side of the square GF or GH $=$ DI $= x$; then will CI $=$ CD $-$ DI $= a - x$.

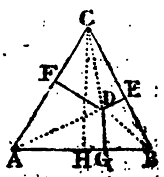


Then, because the like lines in the similar triangles ABC, GFC, are proportional (by theor. 84, Geom.), AB : CD : : GF : CI, that is, $b : a :: x : a - x$. Hence, by multiplying extremes and means, $ab - bx = ax$, and transposing bx, gives $ab = ax + bx$; then dividing by $a + b$, gives $x = \frac{ab}{a + b} =$ GF or GH the side of the inscribed square: which therefore is of the same magnitude, whatever the species or the angles of the triangles may be.

PROBLEM V.

In an Equilateral Triangle, having given the lengths of the three Perpendiculars, drawn from a certain Point within, on the three Sides; to determine the Sides.

LET ABC represent the equilateral triangle, and DE, DF, DG, the given perpendiculars from the point D. Draw the lines DA, DB, DC, to the three angular points; and let fall the perpendicular CH on the base AB. Put the three given perpendiculars DE $= a$, DF $= b$, DG $= c$, and put $x =$ AH or BH, half the side of the equilateral triangle. Then is AC or BC $= 2x$, and by right-angled triangles the perpendicular CH $= \sqrt{AC^2 - AH^2} = \sqrt{4x^2 - x^2} = \sqrt{3x^2} = x\sqrt{3}$.



Now, since the area or space of a rectangle, is expressed by the product of the base and height (cor. 2, th. 81 Geom.), and that a triangle is equal to half a rectangle of equal base and height (cor. 1, th. 26), it follows that,

the whole triangle ABC is $= \frac{1}{2}\text{AB} \times \text{CH} = x \times x\sqrt{3} = x^2\sqrt{3}$,
the triangle ABD $= \frac{1}{2}\text{AB} \times \text{DG} = x \times c = cx$,
the triangle BCD $= \frac{1}{2}\text{BC} \times \text{DE} = x \times a = ax$,
the triangle ACD $= \frac{1}{2}\text{AC} \times \text{DF} = x \times b = bx$.

But the three last triangles make up, or are equal to, the whole former, or great triangle;
that is, $x^2\sqrt{3} = ax + bx + cx$; hence, dividing by x, gives
$x\sqrt{3} = a + b + c$, and dividing by $\sqrt{3}$, gives
$x = \frac{a + b + c}{\sqrt{3}}$, half the side of the triangle sought.

Also, since the whole perpendicular CH is $= x\sqrt{3}$, it is therefore $= a + b + c$. That is, the whole perpendicular CH, is just equal to the sum of all the three smaller perpendiculars DE + DF + DG taken together, wherever the point D is situated.

PROBLEM VI.

In a Right-angled Triangle, having given the Base (3), and the Difference between the Hypothenuse and Perpendicular (1); to find both these two Sides.

PROBLEM VII.

In a Right-angled Triangle, having given the Hypothenuse (5), and the Difference between the Base and Perpendicular (1); to determine both these two Sides.

PROBLEM VIII.

Having given the Area, or Measure of the Space, of a Rectangle, inscribed in a given Triangle; to determine the Sides of the Rectangle.

PROBLEM IX.

In a Triangle, having given the Ratio of the two Sides, together with both the Segments of the Base, made by a Perpendicular from the Vertical Angle; to determine the Sides of the Triangle,

PROBLEM X.

In a Triangle, having given the Base, the Sum of the other two Sid[illegible] Length of a Line drawn from the

Vertical

Vertical Angle to the Middle of the Base; to find the sides of the Triangle.

PROBLEM XI.

In a Triangle, having given the two Sides about the Vertical Angle, with the Line bisecting that Angle, and terminating in the Base; to find the Base.

PROBLEM XII.

To determine a Right-angled Triangle; having given the Lengths of two Lines drawn from the acute angles, to the Middle of the opposite Sides.

PROBLEM XIII.

To determine a Right-angled Triangle; having given the Perimeter, and the Radius of its Inscribed Circle.

PROBLEM XIV.

To determine a Triangle; having given the Base, the Perpendicular, and the Ratio of the two Sides.

PROBLEM XV.

To determine a Right-angled Triangle; having given the Hypothenuse, and the Side of the Inscribed Square.

PROBLEM XVI.

To determine the Radii of three Equal Circles, described in a given Circle, to touch each other and also the Circumference of the given Circle.

PROBLEM XVII.

In a Right-angled Triangle, having given the Perimeter, or Sum of all the Sides, and the Perpendicular let fall from the Right Angle on the Hypothenuse; to determine the Triangle, that is, its Sides.

PROBLEM XVIII.

To determine a Right-angled Triangle; having given the Hypothenuse, and the Difference of two Lines drawn from the two acute angles to the Centre of the Inscribed Circle.

PROBLEM

PROBLEM XIX.

To determine a Triangle; having given the Base, the Perpendicular, and the Difference of the two other Sides.

PROBLEM XX.

To determine a Triangle; having given the Base, the Perpendicular, and the Rectangle or Product of the two Sides.

PROBLEM XXI.

To determine a Triangle; having given the Lengths of three Lines drawn from the three Angles, to the Middle of the opposite Sides.

PROBLEM XXII.

In a Triangle, having given all the three Sides; to find the Radius of the Inscribed Circle.

PROBLEM XXIII.

To determine a Right-angled Triangle; having given the Side of the Inscribed Square, and the Radius of the Inscribed Circle.

PROBLEM XXIV.

To determine a Triangle, and the Radius of the Inscribed Circle; having given the Lengths of three Lines drawn from the three Angles, to the Centre of that Circle.

PROBLEM XXV.

To determine a Right-angled Triangle; having given the Hypothenuse, and the Radius of the Inscribed Circle.

PROBLEM XXVI.

To determine a Triangle; having given the Base, the Line bisecting the Vertical Angle, and the Diameter of the Circumscribing Circle.

LOGARITHMS

OF THE

NUMBERS

FROM

1 to 1000.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0·000000	26	1·414973	51	1·707570	76	1·880814
2	0·301030	27	1·431364	52	1·716003	77	1·886491
3	0·477121	28	1·447158	53	1·724276	78	1·892095
4	0·602060	29	1·462398	54	1·732394	79	1·897627
5	0·698970	30	1·477121	55	1·740363	80	1·903090
6	0·778151	31	1·491362	56	1·748188	81	1·908485
7	0·845098	32	1·505150	57	1·755875	82	1·913814
8	0·903090	33	1·518514	58	1·763428	83	1·919078
9	0·954243	34	1·531479	59	1·770852	84	1·924279
10	1·000000	35	1·544068	60	1·778151	85	1·929419
11	1·041393	36	1·556303	61	1·785330	86	1·934498
12	1·079181	37	1·568202	62	1·792392	87	1·939519
13	1·113943	38	1·579784	63	1·799341	88	1·944483
14	1·146128	39	1·591065	64	1·806180	89	1·949390
15	1·176091	40	1·602060	65	1·812913	90	1·954243
16	1·204120	41	1·612784	66	1·819544	91	1·959041
17	1 230449	42	1·623249	67	1·826075	92	1·963788
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20	1·301030	45	1·653213	70	1·845098	95	1·977724
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22	1·342423	47	1·672098	72	1·857333	97	1·986772
23	1·361728	48	1·681241	73	1·863323	98	1·991226
24	1·380211	49	1·690196	74	1·869232	99	1·995635
25	1·397940	50	1·698970	75	1·875061	100	2·000000

N. B. In the following table, in the last nine columns of each page, where *the first or leading figures change from 9's to 0's, large dots are now introduced instead of the 0's through the rest of the line, to catch the eye, and to indicate that from thence the corresponding natural number in the first column stands in the next lower line, and its annexed first two figures of the Logarithm in the second column.*

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9384	9789	•195	•600	1004	1408	1812	2216	2619	3021
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5336	5640	5943	6246	6549	6852	7154	7457	7759	8061
8362	8664	8965	9266	9567	9868	•168	•469	•769	1068
161368	1667	1967	2266	2564	2863	3161	3460	3758	4055
4353	4650	4947	5244	5541	5838	6134	6430	6726	7022
7317	7613	7908	8203	8497	8792	9086	9380	9674	9968
170262	0555	0848	1141	1434	1726	2019	2311	2603	2895
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152	181844	2129	2415	2700	2985	3270	3555	3839	4123	4407
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155	190332	0612	0892	1171	1451	1730	2010	2289	2567	2846
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157	5899	6176	6453	6729	7005	7281	7556	7832	8107	8382
158	8657	8932	9206	9481	9755	••29	•303	•577	•850	1124
159	201397	1670	1943	2216	2488	2761	3033	3305	3577	3848
160	4120	4391	4663	4934	5204	5475	5746	6016	6286	6556
161	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247
162	9515	9783	••51	•319	•586	•853	1121	1388	1654	1921
163	212188	2454	2720	2986	3252	3518	3783	4049	4314	4579
164	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221
165	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846
166	220108	0370	0631	0892	1153	1414	1675	1936	2196	2456
167	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051
168	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630
169	7887	8144	8400	8657	8913	9170	9426	9682	9938	•193
170	230449	0704	0960	1215	1470	1724	1979	2234	2488	2742
171	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276
172	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795
173	8046	8297	8548	8799	9049	9299	9550	9[illegible]00	••50	•300
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178	250420	0664	0908	1151	1395	1638	1881	2125	2368	2610
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180	5273	5514	5755	5996	6237	6477	6718	6958	7198	7439
181	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833
182	260071	0310	0548	0787	1025	1263	1501	1739	1976	2214
183	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582
184	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937
185	7172	7406	7641	7875	8110	8344	8580	8812	9046	9279
186	9513	9746	9980	•213	•446	•679	•912	1144	1377	1609
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188	4158	4389	4620	4850	5031	5311	5542	5772	6002	6232
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190	8754	8982	9211	9439	9667	9895	•123	•351	•578	•806
191	281033	1261	1488	1715	1942	2169	2396	2622	2849	3075
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193	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578
194	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812
195	290035	0257	0480	0702	0925	1147	1369	1591	1813	2034
96	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246
7	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446
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202	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282
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207	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854
208	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938
209	320146	0354	0562	0769	0977	1184	1391	1598	1805	2012
210	2219	2426	2633	2839	3046	3252	3458	3665	3871	4077
211	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131
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213	8380	8583	8787	8991	9194	9398	9601	9805	...8	.211
214	330414	0617	0819	1022	1225	1427	1630	1832	2034	2236
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216	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260
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221	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157
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224	350248	0442	0636	0829	1023	1216	1410	1603	1796	1989
225	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916
226	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834
227	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744
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231	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301
232	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169
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234	9216	9401	9587	9772	9958	.143	.328	.513	.698	.883
235	371068	1253	1437	1622	1806	1991	2175	2360	2544	2728
236	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565
237	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394
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241	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636
242	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428
243	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212
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248	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025
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254	4834	5005	5176	5346	5517	5688	5858	6029	6199
255	6540	6710	6881	7051	7221	7391	7561	7731	7901
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260	4973	5140	5307	5474	5641	5808	5974	6141	6308
261	6641	6807	6973	7139	7306	7472	7638	7804	7970
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263	9956	.121	.286	.451	.616	.781	.945	1110	1275
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266	4882	5045	5208	5371	5534	5697	5860	6023	6186
267	6511	6674	6836	6999	7161	7324	7486	7648	7811
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273	6163	6322	6481	6640	6800	6957	7116	7275	7433
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277	2480	2637	2793	2950	3106	3263	3419	3576	3732
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279	5604	5760	5915	6071	6226	6382	6537	6692	6848
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281	8706	8861	9015	9170	9324	9478	9633	9787	9941
282	450249	0403	0557	0711	0865	1018	1172	1326	1479
283	1786	1940	2093	2247	2400	2553	2706	2859	3012
284	3318	3471	3624	3777	3930	4082	4235	4387	4540
285	4845	4997	5150	5302	5454	5606	5758	5910	6062
286	6366	6518	6670	6821	6973	7125	7276	7428	7579
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289	460898	1048	1198	1348	1499	1649	1799	1948	2098
290	2398	2548	2697	2847	2997	3146	3296	3445	3594
291	3893	4042	4191	4340	4490	4639	4788	4936	5085
292	5383	5532	5680	5829	5977	6126	6274	6423	6571
293	6868	7016	7164	7312	7460	7608	7756	7904	8052
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3791	3927	4063	4199	4335	4471	4607	4743	4878	5014
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352	6543	6666	6789	6913	7036	7159	7282	7405	7529
353	7775	7898	8021	8144	8267	8389	8512	8635	8758
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355	550228	0351	0473	0595	0717	0840	0962	1084	1206
356	1450	1572	1694	1816	1938	2060	2181	2303	2425
357	2668	2790	2911	3033	3155	3276	3398	3519	3640
358	3883	4004	4126	4247	4368	4489	4610	4731	4852
359	5094	5215	5336	5457	5578	5699	5820	5940	6061
360	6303	6423	6544	6664	6785	6905	7026	7146	7267
361	7507	7627	7748	7868	7988	8108	8228	8349	8469
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363	9907	..26	.146	.265	.385	.504	.624	.743	.863
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366	3481	3600	3718	3837	3955	4074	4192	4311	4429
367	4666	4784	4903	5021	5139	5257	5376	5494	5612
368	5848	5966	6084	6202	6320	6437	6555	6673	6791
369	7026	7144	7262	7379	7497	7614	7732	7849	7967
370	8202	8319	8436	8554	8671	8788	8905	9023	9140
371	9374	9491	9608	9725	9842	9959	.076	.193	.309
372	570543	0660	0776	0893	1010	1126	1243	1359	1476
373	1709	1825	1942	2058	2174	2291	2407	2523	2639
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375	4031	4147	4263	4379	4494	4610	4726	4841	4957
376	5188	5303	5419	5534	5650	5765	5880	5996	6111
377	6341	6457	6572	6687	6802	6917	7032	7147	7262
378	7492	7607	7722	7836	7951	8066	8181	8295	8410
379	8639	8754	8868	8983	9097	9212	9326	9441	9555
380	9784	9898	..12	.126	.241	.355	.469	.583	.697
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384	4331	4444	4557	4670	4783	4896	5009	5122	5235
385	5461	5574	5686	5799	5912	6024	6137	6250	6362
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387	7711	7823	7935	8047	8160	8272	8384	8496	8608
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392	3286	3397	3508	3618	3729	3840	3950	4061	4171
393	4393	4503	4614	4724	4834	4945	5055	5165	5276
394	5496	5606	5717	5827	5937	6047	6157	6267	6377
395	6597	6707	6817	6927	7037	7146	7256	7366	7476
396	7695	7805	7914	8024	8134	8243	8353	8462	8572
397	8791	8900	9009	9119	9228	9337	9446	9556	966
398	9883	9992	.101	.210	.319	.428	.537	.646	.7
399	600973	1082	1191	1299	1408	1517	1625	1734	1

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400	602060	2169	2277	2386	2494	2603	2711	2819	2928	3036
401	3144	3253	3361	3469	3573	3686	3794	3902	4010	4118
402	4226	4334	4442	4550	4658	4766	4874	4982	5089	5197
403	5305	5413	5521	5628	5736	5844	5951	6059	6166	6274
404	6381	6489	6596	6704	6811	6919	7026	7133	7241	7348
405	7455	7562	7669	7777	7884	7991	8098	8205	8312	8419
406	8526	8633	8740	8847	8954	9061	9167	9274	9381	9488
407	9594	9701	9808	9914	. . 21	. 128	. 234	. 341	. 447	. 554
408	610660	0767	0873	0979	1086	1192	1298	1405	1511	1617
409	1723	1829	1936	2042	2148	2254	2360	2466	2572	2678
410	2784	2890	2996	3102	3207	3313	3419	3525	3630	3736
411	3842	3947	4053	4159	4264	4370	4475	4581	4686	4792
412	4897	5003	5108	5213	5319	5424	5529	5634	5740	5845
413	5950	6055	6160	6265	6370	6476	6581	6686	6790	6895
414	7000	7105	7210	7315	7420	7525	7629	7734	7839	7943
415	8048	8153	8257	8362	8466	8571	8676	8780	8884	8989
416	9093	9198	9302	9406	9511	9615	9719	9824	9928	. . 32
417	620136	0240	0344	0448	0552	0656	0760	0864	0968	1072
418	1176	1280	1384	1488	1592	1695	1799	1903	2007	2110
419	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146
420	3249	3353	3456	3559	3663	3766	3869	3973	4076	4179
421	4282	4385	4488	4591	4695	4798	4901	5004	5107	5210
422	5312	5415	5518	5621	5724	5827	5929	6032	6135	6238
423	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263
424	7366	7468	7571	7673	7775	7878	7980	8082	8185	8287
425	8389	8491	8593	8695	8797	8900	9002	9104	9206	9308
426	9410	9512	9613	9715	9817	9919	. . 21	. 123	. 224	. 326
427	630428	0530	0631	0733	0835	0936	1038	1139	1241	1342
428	1444	1545	1647	1748	1849	1951	2052	2153	2255	2356
429	2457	2559	2660	2761	2862	2963	3064	3165	3266	3367
430	3468	3569	3670	3771	3872	3973	4074	4175	4276	4376
431	4477	4578	4679	4779	4880	4981	5081	5182	5283	5383
432	5484	5584	5685	5785	5886	5986	6087	6187	6287	6388
433	6488	6588	6688	6789	6889	6989	7089	7189	7290	7390
434	7490	7590	7690	7790	7890	7990	8090	8190	8290	8389
435	8489	8589	8689	8789	8888	8988	9088	9188	9287	9387
436	9486	9586	9686	9785	9885	9984	. . 84	. 183	. 283	. 382
437	640481	0581	0680	0779	0879	0978	1077	1177	1276	1375
438	1474	1573	1672	1771	1871	1970	2069	2168	2267	2366
439	2465	2563	2662	2761	2860	2959	3058	3156	3255	3354
440	3453	3551	3650	3749	3847	3946	4044	4143	4242	4340
441	4439	4537	4636	4734	4832	4931	5029	5127	5226	5324
442	5422	5521	5619	5717	5815	5913	6011	6110	6208	6306
443	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285
444	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262
445	8360	8458	8555	8653	8750	8848	8945	9043	9140	9237
446	9335	9432	9530	9627	9724	9821	9919	. . 16	. 113	. 210
447	650308	0405	0502	0599	0696	0793	0890	0987	1084	1181
448	1278	1375	1472	1569	1666	1762	1859	1956	2053	2150
449	2246	2343	2440	2536	2633	2730	2826	2923	3019	31[illegible]

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452	5138	5235	5331	5427	5526	5619	5715	5810	5906	6002
453	6098	6194	6290	6386	6482	6577	6673	6769	6864	6960
454	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916
455	8011	8107	8202	8298	8393	8488	8584	8679	8774	8870
456	8965	9060	9155	9250	9346	9441	9536	9631	9726	9821
457	9916	..11	.106	.201	.296	.391	.486	.581	.676	.771
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460	2758	2852	2947	3041	3135	3230	3324	3418	3512	3607
461	3701	3795	3889	3983	4078	4172	4266	4360	4454	4548
462	4642	4736	4830	4924	5018	5112	5206	5299	5393	5487
463	5581	5675	5769	5862	5956	6050	6143	6237	6331	6424
464	6518	6612	6705	6799	6892	6986	7079	7173	7266	7360
465	7453	7546	7640	7733	7826	7920	8013	8106	8199	8293
466	8386	8479	8572	8665	8759	8852	8945	9038	9131	9224
467	9317	9410	9503	9596	9689	9782	9875	9967	..60	.153
468	670246	0339	0431	0524	0617	0710	0802	0895	0988	1080
469	1173	1265	1358	1451	1543	1636	1728	1821	1913	2005
470	2098	2190	2283	2375	2467	2560	2652	2744	2836	2929
471	3021	3113	3205	3297	3390	3482	3574	3666	3758	3850
472	3942	4034	4126	4218	4310	4402	4494	4586	4677	4769
473	4861	4953	5045	5137	5228	5320	5412	5503	5595	5687
474	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602
475	6694	6785	6876	6968	7059	7151	7242	7333	7424	7516
476	7607	7698	7789	7881	7972	8063	8154	8245	8336	8427
477	8518	8609	8700	8791	8882	8973	9064	9155	9246	9337
478	9428	9519	9610	9700	9791	9882	9973	..63	.154	.245
479	680336	0426	0517	0607	0698	0789	0879	0970	1060	1151
480	1241	1332	1422	1513	1603	1693	1784	1874	1964	2055
481	2145	2235	2326	2416	2506	2596	2686	2777	2867	2957
482	3047	3137	3227	3317	3407	3497	3587	3677	3767	3857
483	3947	4037	4127	4217	4307	4396	4486	4576	4666	4756
484	4845	4935	5025	5114	5204	5294	5383	5473	5563	5652
485	5742	5831	5921	6010	6100	6189	6279	6368	6458	6547
486	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440
487	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331
488	8420	8509	8598	8687	8776	8865	8953	9042	9131	9220
489	9309	9398	9486	9575	9664	9753	9841	9930	..19	.107
490	690196	0285	0373	0462	0550	0639	0728	0816	0905	0993
491	1081	1170	1258	1347	1435	1524	1612	1700	1789	1877
492	1965	2053	2142	2230	2318	2406	2494	2583	2671	2759
493	2847	2935	3023	3111	3199	3287	3375	3463	3551	3639
494	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517
495	4605	4693	4781	4868	4956	5044	5131	5219	5307	5394
496	5482	5569	5657	5744	5832	5919	6007	6094	6182	6269
497	6356	6444	6531	6618	6706	6793	6880	6968	7055	7142
498	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014
499	8101	8188	8275	8362	8449	8535	8622	8709	8796	888

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501	9838	9924	..11	..98	.184	.271	.358	.444	.531	.617
502	700704	0790	0877	0963	1050	1136	1222	1309	1395	1482
503	1568	1654	1741	1827	1913	1999	2086	2172	2258	2344
504	2431	2517	2603	2689	2775	2861	2947	3033	3119	3205
505	3291	3377	3463	3549	3635	3721	3807	3893	3979	4065
506	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922
507	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778
508	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632
509	6718	6803	6888	6974	7059	7144	7229	7315	7400	7485
510	7570	7655	7740	7826	7911	7996	8081	8166	8251	8336
511	8421	8506	8591	8676	8761	8846	8931	9015	9100	9185
512	9270	9355	9440	9524	9609	9694	9779	9863	9948	..33
513	710117	0202	0287	0371	0456	0540	0625	0710	0794	0879
514	0963	1048	1132	1217	1301	1385	1470	1554	1639	1723
515	1807	1892	1976	2060	2144	2229	2313	2397	2481	2566
516	2650	2734	2818	2902	2986	3070	3154	3238	3326	3407
517	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246
518	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084
519	5167	5251	5335	5418	5502	5586	5669	5753	5836	5920
520	6003	6087	6170	6254	6337	6421	6504	6588	6671	6754
521	6838	6921	7004	7088	7171	7254	7338	7421	7504	7587
522	7671	7754	7837	7920	8003	8086	8169	8253	8336	8419
523	8502	8585	8668	8751	8834	8917	9000	9083	9165	9248
524	9331	9414	9497	9580	9663	9745	9828	9911	9994	..77
525	720159	0242	0325	0407	0490	0573	0655	0738	0821	0903
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527	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552
528	2634	2716	2798	2881	2963	3045	3127	3209	3291	3374
529	3456	3538	3620	3702	3784	3866	3948	4030	4112	4194
530	4276	4358	4440	4522	4604	4685	4767	4849	4931	5013
531	5095	5176	5258	5340	5422	5503	5585	5667	5748	5830
532	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646
533	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460
534	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273
535	8354	8435	8516	8597	8678	8759	8841	8922	9003	9084
536	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893
537	9974	..55	.136	.217	.298	.378	.459	.540	.621	.702
538	730782	0863	0944	1024	1105	1186	1266	1347	1428	1508
539	1589	1669	1750	1830	1911	1991	2072	2152	2233	2313
540	2394	2474	2555	2635	2715	2796	2876	2956	3037	3117
541	3197	3278	3358	3438	3518	3598	3679	3759	3839	3919
542	3999	4079	4160	4240	4320	4400	4480	4560	4640	4720
543	4800	4880	4960	5040	5120	5200	5279	5359	5439	5519
544	5599	5679	5759	5838	5918	5998	6078	6157	6237	6317
545	6397	6476	6556	6635	6715	6795	6874	6954	7034	7113
546	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908
547	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701
548	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493
549	9572	9651	9731	9810	9889	9968	..47	.126	.205	.28

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551	1152	1230	1309	1388	1467	1546	1624	1703	1782	1860
552	1939	2018	2096	2175	2254	2332	2411	2489	2568	2646
553	2725	2804	2382	2961	3039	3118	3196	3275	3353	3431
554	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215
555	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997
556	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777
557	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556
558	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334
559	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110
560	8188	8266	8343	8421	8498	8576	8653	8731	8808	8885
561	8963	9040	9118	9195	9272	9350	9427	9504	9582	9659
562	9736	9814	9891	9968	..45	.123	.200	.277	.354	.431
563	750508	0586	0663	0740	0817	0894	0971	1048	1125	1202
564	1279	1356	1433	1510	1587	1664	1741	1818	1895	1972
565	2048	2125	2202	2279	2356	2433	2509	2586	2663	2740
566	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506
567	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272
568	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036
569	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799
570	5875	5951	6027	6103	6180	6256	6332	6408	6484	6560
571	6636	6712	6788	6864	6940	7016	7092	7168	7244	7320
572	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079
573	8155	8230	8306	8382	8458	8533	8609	8685	8761	8836
574	8912	8988	9063	9139	9214	9290	9366	9441	9517	9592
575	9668	9743	9819	9894	9970	..45	.121	.196	.272	.347
576	760422	0498	0573	0649	0724	0799	0875	0950	1025	1101
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578	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604
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580	3428	3503	3578	3653	3727	3802	3877	3952	4027	4101
581	4176	4251	4326	4400	4475	4550	4624	4699	4774	4848
582	4923	4998	5072	5147	5221	5296	5370	5445	5520	5594
583	5669	5743	5818	5892	5966	6041	6115	6190	6264	6338
584	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082
585	7156	7230	7304	7379	7453	7527	7601	7675	7749	7823
586	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564
587	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303
588	9377	9451	9525	9599	9673	9746	9820	9894	9968	..42
589	770115	0189	0263	0336	0410	0484	0557	0631	0705	0778
590	0852	0926	0999	1073	1146	1220	1293	1367	1440	1514
591	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248
592	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981
593	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713
594	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444
595	4517	4590	4663	4736	4809	4882	4955	5028	5100	5173
	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902
	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629
	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354
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602	9596	9669	9741	9813	9885	9957	..29	.101	.173	.245
603	780317	0389	0461	0533	0605	0677	0749	0821	0893	0965
604	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684
605	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401
606	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117
607	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832
608	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546
609	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259
610	5330	5401	5472	5543	5615	5686	5757	5828	5899	5970
611	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680
612	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390
613	7460	7531	7602	7673	7744	7815	7885	7956	8027	9098
614	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804
615	8875	8946	9016	9087	9157	9228	9299	9369	9440	9510
616	9581	9651	9722	9792	9863	9933	...4	..74	.144	.215
617	790285	0356	0426	0496	0567	0637	0707	0778	0848	0918
618	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620
619	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322
620	2392	2462	2532	2602	2672	2742	2812	2882	2952	3022
621	3092	3162	3231	3301	3371	3441	3511	3581	3651	3721
622	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418
623	4488	4558	4627	4697	4767	4836	4906	4976	5045	5115
624	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811
625	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505
626	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198
627	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890
628	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582
629	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272
630	9341	9409	9478	9547	9616	9685	9754	9823	9892	9961
631	800029	0098	0167	0236	0305	0373	0442	0511	0580	0648
632	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335
633	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021
634	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705
635	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389
636	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071
637	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753
638	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433
639	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112
640	6180	6248	6316	6384	6451	6519	6587	6655	6723	6790
641	6858	6926	6994	7061	7129	7197	7264	7332	7400	7467
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643	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818
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645	9560	9627	9694	9762	9829	9896	9964	..31	..98	.165
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648	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178
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652	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847
653	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511
654	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175
655	6241	6308	6374	6440	6506	6573	6639	6705	6771	6838
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657	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160
658	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820
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663	1514	1579	1645	1710	1775	1841	1906	1972	2037	2103
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665	2822	2887	2952	3018	3083	3148	3213	3279	3344	3409
666	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061
667	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711
668	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361
669	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010
670	6075	6140	6204	6269	6334	6399	6464	6528	6593	6658
671	6723	6787	6852	6917	6981	7046	7111	7175	7240	7305
672	7369	7434	7499	7563	7628	7692	7757	7821	7886	7951
673	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595
674	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239
675	9304	9368	9432	9497	9561	9625	9690	9754	9818	9882
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678	1230	1294	1358	1422	1486	1550	1614	1678	1742	1806
679	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445
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682	3784	3848	3912	3975	4039	4103	4166	4230	4294	4357
683	4421	4484	4548	4611	4675	4739	4802	4866	4929	4993
684	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627
685	5691	5754	5817	5881	5944	6007	6071	6134	6197	6261
686	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894
687	6957	7020	7083	7146	7210	7273	7336	7399	7462	7525
688	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156
689	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786
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692	840106	0169	0232	0294	0357	0420	0482	0545	0608	0671
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694	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922
695	1985	2047	2110	2172	2235	2297	2360	2422	2484	2547
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697	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793
698	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415
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703	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511
704	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128
705	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743
706	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358
707	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972
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709	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197
710	1258	1320	1381	1442	1503	1564	1625	1686	1747	1809
711	1870	1931	1992	2053	2114	2175	2236	2297	2358	2419
712	2480	2541	2602	2663	2724	2785	2846	2907	2968	3029
713	3090	3150	3211	3272	3333	3394	3455	3516	3577	2637
714	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245
715	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852
716	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459
717	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064
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720	7332	7393	7453	7513	7574	7634	7694	7755	7815	7875
721	7935	7995	8056	8116	8176	8236	8297	8357	8417	8477
722	8537	8597	8657	8718	8778	8838	8898	8958	9018	9078
723	9138	9198	9258	9318	9379	9439	9499	9559	9619	9679
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725	860338	0398	0458	0518	0578	0637	0697	0757	0817	0877
726	0937	0996	1056	1116	1176	1236	1295	1355	1415	1475
727	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072
728	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668
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730	3323	3382	3442	3501	3561	3620	3680	3739	3799	3858
731	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452
732	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045
733	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637
734	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228
735	6287	6346	6405	6465	6524	6583	6642	6701	6760	6819
736	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409
737	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998
738	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586
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744	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098
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748	3902	3960	4018	4076	4134	4192	4250	4308	4366	442[illegible]
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752	6218	6276	6333	6391	6449	6507	6564	6622	6680	6737
753	6795	6853	6910	6968	7026	7083	7141	7199	7256	7314
754	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889
755	7947	8004	8062	8119	8177	8234	8292	8349	8407	8464
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759	880242	0299	0356	0413	0471	0528	0585	0642	0699	0756
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762	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468
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780	2095	2150	2206	2262	2317	2373	2429	2484	2540	2595
781	2651	2707	2762	2818	2873	2929	2985	3040	3096	3151
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783	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261
784	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814
785	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367
786	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920
787	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471
788	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022
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792	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218
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795	900367	0422	0476	0531	0586	0640	0695	0749	0804	0859
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797	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948
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802	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661
803	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202
804	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742
805	5796	5850	6904	5958	6012	6066	6119	6173	6227	6281
806	6335	6389	6443	6497	6351	6604	6658	6712	6766	6820
807	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358
808	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895
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813	910091	0144	0197	0251	0304	0358	0411	0464	0518	0571
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818	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231
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820	3814	3867	3920	3973	4026	4079	4132	4184	4237	4290
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840	4279	4331	4383	4434	4486	4538	4589	4641	4693	4744
841	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261
842	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776
843	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291
844	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805
845	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319
846	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832
847	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345
848	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857
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852	930440	0491	0542	0592	0643	0694	0745	0796	0847	0898
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854	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915
855	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423
856	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930
857	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437
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859	3993	4044	4094	4145	4195	4246	4296	4347	4394	4448
860	4498	4549	4599	4650	4700	4751	4801	4852	4902	4953
861	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457
862	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960
863	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463
864	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966
865	7016	7066	7117	7167	7217	7267	7317	7367	7418	7468
866	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969
867	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470
868	8520	8570	8620	8670	8720	8770	8820	8870	8919	8970
869	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469
870	9519	9569	9619	9669	9719	9769	9819	9869	9918	9968
871	940018	0068	0118	0168	0218	0267	0317	0367	0417	0467
872	0516	0566	0616	0666	0716	0765	0815	0865	0915	0964
873	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462
874	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958
875	2008	2058	2107	2157	2207	2256	2306	2355	2405	2455
876	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950
877	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445
878	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939
879	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433
880	4483	4532	4581	4631	4680	4729	4779	4828	4877	4927
881	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419
882	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912
883	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403
884	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894
885	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385
886	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875
887	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364
888	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853
889	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341
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894	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775
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896	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744
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898	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711
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908	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516
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942	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466
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947	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763
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961	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130
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968	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279
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971	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622
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973	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514
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980	1226	1270	1315	1359	1403	1448	1492	1536	1580	1625
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993	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343
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995	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216
996	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652
997	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087
998	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522
999	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957

END OF VOL. I